On Harder-Narasimhan stratifications

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Bundles and semistability

- The concept of **semistability** is defined for many different kinds of algebro-geometric ‘bundles’, such as vector bundles, coherent pure sheaves, Higgs sheaves, connections, principal bundles with a reductive structure group G, and so on.
- The semistable objects are better behaved in many ways than those which are **unstable** (which means not semistable). e.g. Narasimhan-Seshadri theorem, boundedness, good moduli spaces.
- Harder and Narasimhan (in mid 1970s) initiated a method to understand any unstable object in terms of a natural **‘filtration’** of it by semistable objects which are ‘smaller’ than the original object. For a principal bundle, this takes the form of a reduction of its structure group G to a parabolic subgroup of G.
- Such a filtration is called the Harder-Narasimhan filtration (**HN-filtration**) or the **canonical reduction** of the object.

Families of bundles $E/X/S$

- Any 'bundle' (vector bundle, coherent sheaf, Higgs sheaf, principal bundle etc) E on a projective variety $(X, \mathcal{O}_X(1))$ over a field K has a discrete combinatorial or 'topological' type (given by its rank, degree, Hilbert polynomial, characteristic class, etc). This type is locally constant over the parameter space in any **family** E of bundles over a family $X \to S$ of spaces parameterized by a space S with a given relatively ample line bundle $\mathcal{O}_{X/S}(1)$ on X.
- A combinatorial ('topological') type can be attached to the HN-filtration or canonical reduction of a bundle $E/X/K$, called its **Harder-Narasimhan type** $\mathrm{HN}(E)$.
- For a given kind of bundles (say, principal G-bundles where G is a given reductive group), all possible HN-types of bundles form a partially ordered set.
- The semistable bundles have the minimum HN-type τ_0 among all bundles of a given 'topological' type on a projective variety $(X/K, \mathcal{O}_{X/K}(1))$.

Variation of HN-type in a family

- How does the type $\mathrm{HN}(E_s)$ of $E_s = E|X_s$ vary in a family $E/X/S$ of bundles as s varies over the parameter space S?
- The points $s \in S$ such that $E_s = E|X_s$ is semistable form an open subscheme S^{ss} (possibly empty) of the parameter space S.
- The locus where $\mathrm{HN}(E_s) = \tau_0$ is S^{ss}, the open subscheme of S of semistable bundles.
- For any HN-type τ, the Zariski closure of S^{τ} (the subset of S defined by $\mathrm{HN}(E_s) = \tau$) is contained in $S^{\geq\tau}$ (the subset of S defined by $\mathrm{HN}(E_s) \geq \tau$).
- The subsets $S^{\geq\tau}$ and $S^{>\tau}$ are closed in S. In particular, S^{τ} is a locally closed subset of S.
- For any family of bundles $E/X/S$, the set-theoretic disjoint union $S = \cup S^{\tau}$ can be called the **(set-theoretic) Harder-Narasimhan stratification** of S.

From sets to schemes: basic questions

- The above facts were known for various kinds of bundles for a few decades, due to S.S. Shatz and others, before I raised the following questions in 2008. These were inspired by my collaboration with Leticia Brambila-Paz.
- For any family $E/X/S$ and an HN-type τ, does the locally closed subset $S^\tau = \{s \in S \,|\, \mathrm{HN}(E_s) = \tau\}$ have a **natural** structure of a scheme, with an appropriate **universal property**?
- What should be its **functor of points**?
- With its functorial scheme structure, will S^τ be a **locally closed subscheme** of S?
- Is there a natural **algebraic stack** of bundles of HN-type τ? What are its properties?
- Brambila-Paz, Kirwan and various others (including myself) have worked on **moduli spaces** for fixed HN-types. I will not address this question here.

References for this talk

[N] arXiv:0909.0891 : Harder-Narasimhan stacks are constructed for **coherent pure sheaves of $\mathcal{O}$-modules** on the fibers of a projective scheme over a noetherian base scheme (of possibly mixed characteristic).

[G-N-1] arXiv:1208.5572 (jointly with Sudarshan Gurjar) : Harder-Narasimhan stacks are constructed for Λ**-modules** in the above set-up (in particular, for **Higgs sheaves, connections**, etc). This paper also treats **principal G-bundles** over **curves in characteristic zero**, where G is a reductive group (which turned out to be already proved by Kai Behrend in his thesis).

[G-N-2] arXiv:1505.02236 (jointly with Gurjar) : This treats principal G-bundles in **arbitrary dimensions, in characteristic zero**.

[G-N-3] arXiv:1605.08997v4 Nov 2020 (jointly with Gurjar) : This treats principal G-bundles in **arbitrary dimensions, in any characteristic**. This is the focus of the rest of this talk.

Some basics about reductive groups

Here onwards, we will use some facts from the theory of reductive groups. The 2016 textbook of J.S. Milne is an excellent reference.

- A **split torus** over is linear algebraic group T over a field k that is isomorphic to $\mathbb{G}_m^n$ over k ($\mathbb{G}_m = GL_1 = \mathrm{Spec}\, k[t, t^{-1}]$). A **torus** is linear algebraic group T over a field k which becomes isomorphic to $\mathbb{G}_m^n$ on base-change to an extension field of k.
- A connected reduced linear algebraic group over a field k is called **reductive** if all its connected closed reduced normal solvable subgroups are tori. Examples: GL_n, SL_n, SO_n, Sp_n are reductive for all $n \geq 0$. The product of two reductive groups is reductive.
- A **split reductive group** over k is a pair (G, T) where G is reductive, and T is a maximal torus contained in G (as a closed subgroup), and moreover T is a split torus over k.
- Given such a (G, T), we can choose a **Borel subgroup** B (a maximal closed reduced solvable subgroup) such that $T \subset B \subset G$.

Roots, weights, Weyl chamber, partial order, etc.

We fix the following notation:

- G a reductive group scheme over a base field k.
- $T \subset B \subset G$ a maximal torus and a Borel. We assume that T is split over k.
- $\Delta \subset X^*(T)$ corresponding set of all simple roots.
- $\omega_\alpha \in \mathbb{Q} \otimes X^*(T)$ dominant weight corresponding to $\alpha \in \Delta$.
- $I_P \subset \Delta$ the set of inverted simple roots corresponding to any standard parabolic $B \subset P \subset G$.
- $\overline{C} \subset \mathbb{Q} \otimes X_*(T)$ closed positive Weyl chamber. Recall that $\overline{C} = \{\mu \in \mathbb{Q} \otimes X_*(T) \,|\, \langle \alpha, \mu \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$.
- A partial order on $\overline{C}$ defined by putting $\mu \leq \nu$ if $\langle \omega_\alpha, \mu \rangle \leq \langle \omega_\alpha, \nu \rangle$ for all $\alpha \in \Delta$ and $\chi(\mu) = \chi(\nu)$ for all $\chi \in \widehat{G} = Hom(G, \mathbb{G}_m)$.

Principal bundles, semistability -1

- We fix a base field k, and a split-reductive group G over k, and subgroups $T \subset B \subset G$ where T is a maximal split torus and B a Borel subgroup.
- Given an extension field K of k, we will write G_K, T_K, B_K and P_K where $P \supset B$ simply as G, T, B, P when no confusion can result. These P_K are exactly the parabolics of G_K that contain B_K.
- For any principal G-bundle E (means G_K-bundle) over a projective variety $(X, \mathcal{O}_X(1))$ over an extension field K of k, the notion of semistability was defined in Ramanathan's thesis ($\sim$1975).
- The bundle E is assumed to be locally trivial in the étale topology (but not necessarily so in the Zariski topology).
- Semistability makes sense even for bundles defined on **big** open subsets $U \subset X$ ('big' means $\mathrm{dim}(X - U) \leq \mathrm{dim}(X) - 2$). This is important for defining a canonical reduction of E.

Principal bundles, semistability -2

- A closed reduced subgroup $P \subset G$ is called a **parabolic subgroup** if $B \subset P$ for some Borel B, equivalently, if the variety G/P is complete ('compact').
- Vector bundle corresponds to a principal GL_n-bundles and a subbundle correspond to a **reduction of structure group** from $G = GL_n$ to a parabolic subgroup $P \subset G$. The definition of semistability of vector bundles has a translation in terms of such reductions, and this becomes the general definition of semistability for principal bundles.
- An unavoidable complication when $\dim(X) \geq 2$ is that we must consider such reductions that are possibly only defined on a **big** open subset $U \subset X$, where 'big' means $codim(X - U, X) \geq 2$. The resulting notion of semistability for principal G-bundles corresponds to the notion of **slope-semistability** for vector bundles ($rank(E)deg(F) - rank(F)deg(E) \leq 0$).

Principal bundles, semistability -3

- A principal G-bundle E defined on a big open subset U of a geometrically irreducible smooth projective variety $(X, \mathcal{O}_X(1))$ over a field extension K/k is **semistable** if the following holds.
- For any rational reduction σ of the structure group G to a standard parabolic P and any **dominant character** $\chi : P \to \mathbb{G}_m$, the line bundle $\chi_*\sigma^*E$ on X has degree ≤ 0. Here, σ^*E is the principal P-bundle defined by σ, and χ associates to it the line bundle $\chi_*\sigma^*E$.
- Dominance of χ means for any positive co-root $\alpha^\vee$ of (G, B, T), we have $\langle \chi, \alpha^\vee \rangle > 0$ and χ is trivial on $Z_0(G)$, the connected component of the center of G.
- The line bundle $\chi_*\sigma^*E$ is defined on a big open subset of X, hence has a uniquely unique prolongation to X, by smoothness of X/K using Weil divisors and their closures. So the degree of $\chi_*\sigma^*E$ makes sense.

The type of a rational parabolic reduction

- To each rational parabolic reduction σ of E to a standard parabolic P, there is an associated **type** $\mu_{(P,\sigma)}(E) \in \mathbb{Q} \otimes X_*(T)$ (a virtual 1-PS of T), defined as follows, where χ varies over $X^*(T)$ ($= \widehat{T}$, the group of all homomorphisms $T \to \mathbb{G}_m$).

$$\langle \chi, \mu_{(P,\sigma)}(E) \rangle = \begin{cases} \deg(\chi_* \sigma^* E) & \text{if } \chi \in \widehat{P}|_T, \\ 0 & \text{if } \chi \in I_P. \end{cases}$$

- In the above, $\widehat{P}$ is the group of all homomorphisms $P \to \mathbb{G}_m$ and I_P is the subset of the base Δ of roots of (G, B, T) that are inverted to define $P \supset B$. Note that $\mathbb{Q} \otimes X^*(T) = \mathbb{Q}\widehat{P}|_T \oplus \mathbb{Q} I_P$, so this uniquely specifies the type $\mu_{(P,\sigma)}$.

Canonical reduction

- A **canonical reduction** (P, σ) of E is a rational reduction $\sigma : U \to E/P$ to a standard parabolic P such that:

- If $\rho : P \to L$ is the Levi quotient of P (by its unipotent radical) then $\rho_* \sigma^* E$ is a semistable principal L-bundle defined on the big open subscheme U (note that the definition of semistability makes sense also for a principal bundle defined on any big open subscheme of X).

- For any non-trivial character $\chi \in \widehat{P}$ whose restriction to the chosen maximal torus $T \subset B \subset P$ has the form $\sum n_i \alpha_i$ with $\alpha_i \in \Delta$ where $n_i \geq 0$, we must have $\deg(\chi_* \sigma^* E) > 0$.

The condition $(*)$

- We will henceforth assume that the various Levi quotients H of G (including G itself) have the following property $(*)$.
- $(*)$ Given any field extensions $L/K/k$, a geometrically connected smooth projective variety $(X, \mathcal{O}_{X/K}(1))$ over K and a principal H-bundle E defined on a big open subset in X, the base change E_L on $(X_L, \mathcal{O}_{X_L/L}(1))$ is semistable if and only if E is semistable.
- By the **semistable restriction theorem of Gurjar**, it is enough to assume this when $\dim(X) = 1$.
- The condition $(*)$ is satisfied by all classical reductive groups. The exceptional simple groups satisfy $(*)$ in large enough characteristics. The Behrend conjecture implies $(*)$. J. Heinloth (in IMRN 2008) has more on this.

Harder-Narasimhan type

- **Theorem of Behrend**: Given a principal G-bundle $E/X/K$, there exists a unique canonical reduction (P,σ). Its type $\mu_{(P,\sigma)}$ is called the **Harder-Narasimhan type**, denoted by $\mathrm{HN}(E)$.

- We have $\mathrm{HN}(E) \in \overline{C} \subset \mathbb{Q} \otimes X_*(T)$. In fact, $\langle \alpha, \mathrm{HN}(E)\rangle > 0$ for all $\alpha \in \Delta - I_P$ and $\langle \beta, \mathrm{HN}(E)\rangle = 0$ for all $\beta \in I_P$, so it lies in the face of $\overline{C}$ corr. to P. The open face corr. to $P = B$, and the vertex $0 \in \overline{C}$ corr. to $P = G$, which is case when E is semistable.

- **Maximality** Given any other rational parabolic reduction (Q,τ) of E, we have
$$\mu_{(Q,\tau)} \leq \mathrm{HN}(E)$$
w.r.t. the natural partial ordering on $\overline{C}$.

- Other proofs by Biswas-Holla and Gurjar-Nitsure. Apparently Ramanathan knew the theorem but did not publish any proof.

Definition of a relative rational P-reduction for $E/X/S$

- For $P \supset B$, let $V_P = H^0(G/P, \omega_{G/P}^{-1})^\vee$ as a G-representation. Then we have a G-equivariant projective embedding $G/P \hookrightarrow \mathbf{P}(V)$ as $\omega_{G/P}^{-1}$ is very ample. The P-fixed point $eP \in G/P$ corresponds to a line $J \subset V_P$ which is a P-representation via a dominant character $\lambda_P : P \to \mathbb{G}_m$.

- A **relative rational *P*-reduction** of E over an S-scheme T is a reduction $\sigma : U \to E_T/P$ where $U \subset X_T$ is a **relatively big** open subset (means each fiber U_t is big in X_t), such the associated line bundle $(\lambda_P)_*\sigma^*(E_T)$ on U has a prolongation to a line bundle L on X_T (such a prolongation is uniquely unique, if it exists).

- If $T = \mathrm{Spec}\, K$ for any field K over S, then note that the above definition reduces to the usual definition of a rational P-reduction over X_K, as in that case any line bundle admits a prolongation.

The groupoid $Bun^{\tau}_{X/S}(G)$ over S

- Let k be a field, (G, T) a split reductive group over k, and B a chosen Borel with $T \subset B \subset G$.

- Let $(X/S, \mathcal{O}_{X/S}(1))$ be a smooth family of geometrically irreducible projective varieties over a noetherian k-scheme S. For any S-scheme T let $Bun_{X/S}(G)(T)$ denote the groupoid of all principal G-bundles on X_T. Then (as proved e.g. in [G-N-3]) $Bun_{X/S}(G)$ is an algebraic stack over S.

- Moreover, it is shown in [G-N-3] that if G satisfies the condition $(*)$, then for any $\tau \in \overline{C}$ (the closed positive Weyl chamber for G), the algebraic stack $Bun_{X/S}(G)$ has open substacks $Bun^{\not> \tau}_{X/S}(G)(T)$ and $Bun^{\leq \tau}_{X/S}(G)(T)$. In particular, semistable G-bundles on fibers of X/S form an open substack of $Bun_{X/S}(G)$.

Main Theorem

- For any $\tau \in \overline{C}$ (the closed positive Weyl chamber for G), let $Bun^{\tau}_{X/S}(G)(T)$ be the groupoid of all principal G-bundles $E/X_T/T$ together with a relative canonical reduction of type τ. The following is the main result of [G-N-3], assuming G satisfies $(*)$.

- **Theorem** *$Bun^{\tau}_{X/S}(G)$ is an algebraic stack over S. The forgetful 1-morphism $Bun^{\tau}_{X/S}(G) \to Bun_{X/S}(G)$ is schematic, of finite type, separated, universally injective (radicial) and induces an isomorphism on all residue fields.*
  The forgetful 1-morphism $Bun^{\tau}_{X/S}(G) \to Bun^{\not>\tau}_{X/S}(G)$ (the open substack of $Bun_{X/S}(G)$ defined by $\mathrm{HN}(E) \not> \tau$) is finite. Moreover, if the Behrend conjecture holds for G then $Bun^{\tau}_{X/S}(G) \to Bun^{\not>\tau}_{X/S}(G)$ is a closed embedding.

A glimpse of the proof

- A rational parabolic reduction of $E/X/S$ of type τ corresponds to the equivalence class of a pair (L, i), where L is a line bundle on X and $i : L \to E(V_P)$ is a homomorphism satisfying various properties.
- The equivalence classes of pairs (L, i) can be parameterized by using the projective scheme $Proj(Q)$ of a certain Grothendieck Q-sheaf over the relative Picard scheme $Pic_{X/S}$. We expect the use $Proj(Q)$ to be of independent interest.
- Then various standard tools for dealing with coherent sheaves on projective schemes over noetherian bases such as flat descent, semicontinuity and base change, flattening stratifications, Castelnuovo-Mumford regularity, Hilbert and Quot schemes, relative duality, cohomology vanishing result of the Enriques-Severi-Zariski kind, etc. combined with basic facts about reductive groups and their representations, prove the results.

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Thank you for your kind attention.