Density function for the second coefficients of the Hilbert-Kunz function

Mandira Mondal
(partially) based on joint work with Prof. V. Trivedi

Chennai Mathematical Institute
Chennai, India

A Conference on topics in Algebraic Geometry and Commutative Algebra
July, 2022

Outline

1. Hilbert-Kunz function
2. Hilbert-Kunz density function
3. Projective toric variety
4. Density function for the second coefficient

Hilbert-Kunz function and Hilbert-Kunz multiplicity

- $(R, \mathbf{m})$, a Noetherian local ring, $\dim(R) = d$, $char(R) = p > 0$
- I an $\mathbf{m}$-primary ideal
- M a finitely generated R-module.
- $n \in \mathbb{N}$ and $q = p^n$.
- $I^{[q]} =$ ideal generated by q-th power of elements of I
- The **Hilbert-Kunz function** of M with respect to the ideal I, is defined as

$$\mathrm{HK}(M, I)(n) = \ell(M/I^{[q]}M).$$

Hilbert-Kunz function and Hilbert-Kunz multiplicity

- $(R, \mathbf{m})$, a Noetherian local ring, $\dim(R) = d$, $char(R) = p > 0$
- I an $\mathbf{m}$-primary ideal
- M a finitely generated R-module.
- $n \in \mathbb{N}$ and $q = p^n$.
- $I^{[q]}$ = ideal generated by q-th power of elements of I
- The **Hilbert-Kunz function** of M with respect to the ideal I, is defined as
$$\mathrm{HK}(M, I)(n) = \ell(M/I^{[q]}M).$$
- (Monsky '83) There exists a real constant $e_{HK}(M, I) \geq 1$, such that
$$\mathrm{HK}(M, I)(n) = e_{HK}(M, I)q^d + O(q^{(d-1)}).$$
- The Hilbert-Kunz multiplicity of M with respect to the ideal I is defined to be
$$e_{HK}(M, I) := \lim_{n \to \infty} \frac{\ell(M/I^{[q]}M)}{q^d}.$$

Hilbert-Kunz function and Hilbert-Kunz multiplicity

- $(R, \mathbf{m})$, a Noetherian local ring, $\dim(R) = d$, $char(R) = p > 0$
- I an $\mathbf{m}$-primary ideal
- M a finitely generated R-module.
- $n \in \mathbb{N}$ and $q = p^n$.
- $I^{[q]} =$ ideal generated by q-th power of elements of I
- The **Hilbert-Kunz function** of M with respect to the ideal I, is defined as
$$\mathrm{HK}(M, I)(n) = \ell(M/I^{[q]}M).$$
- (Monsky '83) There exists a real constant $e_{HK}(M, I) \geq 1$, such that
$$\mathrm{HK}(M, I)(n) = e_{HK}(M, I)q^d + O(q^{(d-1)}).$$
- The Hilbert-Kunz multiplicity of M with respect to the ideal I is defined to be
$$e_{HK}(M, I) := \lim_{n \to \infty} \frac{\ell(M/I^{[q]}M)}{q^d}.$$

Properties of the HK multiplicity

- Let $e_0(R, I)$ be the Hilbert-Samuel multiplicity of R with respect to the ideal I.
$$\frac{e_0(R, I)}{d!} \leq e_{HK}(R, I) \leq e_0(R, I).$$
- (Additive property) Moreover, e_{HK} is additive:
$$e_{HK}(M, I) = \sum_{\{P \in \mathrm{Spec}(R) \mid \dim(R/P) = \dim(R)\}} e_{HK}(R/P, I)\ell_{R_P}(M_P).$$
- (Watanabe-Yoshida) Let $(R, \mathbf{m})$ be a formally unmixed local ring of prime characteristic $p > 0$. If $e_{HK}(R, \mathbf{m}) = 1$, then R is regular.

e_{HK} and tight closure

- Let R^0 be the complement of union of all minimal primes of R.
- Let I be an ideal of R. The *tight closure* of I (denoted by I^*), is defined as

$$I^* = \{x \in R \mid \text{ there exists } c \in R^0 \text{ such that } cx^q \in I^{[q]} \text{ for all large } n\}$$

- (Hochster-Huneke) Let $J \subseteq I$ be $\mathbf{m}$-primary ideals of the ring R. Then

$$e_{HK}(R, J) = e_{HK}(R, I) \text{ if } J^* = I^*.$$

The converse is true when R is formally unmixed.

Examples

Let C be an irreducible cubic curve in $\mathbb{P}^2$. Let $(R, \mathbf{m})$ be the associated homogeneous coordinate ring. For $n \in \mathbb{N}$, we write $q = p^n$.

- C is a cuspidal cubic curve. Then

$$\mathrm{HK}(R, \mathbf{m})(q) = \begin{cases} \frac{7}{3}q^2 & \text{if } p = 3 \\ \frac{7}{3}q^2 - \frac{4}{3} & \text{if } p \neq 3 \end{cases}$$

- C is a nodal cubic curve. Then

$$\mathrm{HK}(R, \mathbf{m})(q) = \begin{cases} \frac{7}{3}q^2 - \frac{1}{3}q - 1 & \text{if } q \neq 2 \bmod 3 \\ \frac{7}{3}q^2 - \frac{1}{3}q - \frac{5}{3} & \text{if } q = 2 \bmod 3 \end{cases}$$

- C is an elliptic curve and $p \neq 2$. Then $\mathrm{HK}(R, \mathbf{m})(q) = \frac{9}{4}q^2 - \frac{5}{4}$.

A map on class group of R

- $(R, \mathbf{m})$, an excellent normal Noetherian local domain, $\dim(R) = d$, $char(R) = p$ with perfect residue field; $I \subset R$, an $\mathbf{m}$-primary ideal.
- M, a finitely generated torsion free R-module.
- (Huneke-McDermott-Monsky '04) The limit

$$\lim_{q \to \infty} \frac{1}{q^{d-1}} [\ell(M/I^{[q]}M) - \mathrm{rank}(M)\ell(R/I^{[q]})]$$

  exists. We denote this number by $\tau(M, I)$.
- We have $\tau(M, I) = \tau(N, I)$ for two modules M and N of R, if they have same class in the class group of R.

The second coefficient

- (Huneke-McDermott-Monsky '04) Let ${}^1R$ be the R-module R via the action of the Frobenius map $F : R \longrightarrow R$. Let

$$\beta(M, I) = \tau(M, I) - \frac{\text{rank}(M)}{p^d - p^{d-1}} \tau({}^1R, I).$$

Then

$$\text{HK}(M, I)(n) = e_{HK}(M, I)q^d + \beta(M, I)q^{d-1} + O(q^{d-2}).$$

- (Chan-Kurano '16, Hochster-Yao '09) The second coefficient exists for normal rings satisfying (R_1) condition.
- (Bruns-Gubeladze '09)The HK function of a normal affine monoid is a quasipolynomial and the second coefficient is constant.

The second coefficient

- (Huneke-McDermott-Monsky '04) Let ${}^1R$ be the R-module R via the action of the Frobenius map $F : R \longrightarrow R$. Let

$$\beta(M, I) = \tau(M, I) - \frac{\text{rank}(M)}{p^d - p^{d-1}} \tau({}^1R, I).$$

Then

$$\text{HK}(M, I)(n) = e_{HK}(M, I)q^d + \beta(M, I)q^{d-1} + O(q^{d-2}).$$

- (Chan-Kurano '16, Hochster-Yao '09) The second coefficient exists for normal rings satisfying (R_1) condition.
- (Bruns-Gubeladze '09)The HK function of a normal affine monoid is a quasipolynomial and the second coefficient is constant.

e_{HK} may depend on the characteristic.

HK multiplicity has been difficult to compute, even for standard graded rings as many usual techniques for Hilbert-Samuel multiplicity do not work for e_{HK}.

For example

- for a hypersurface, e_{HK} may not be the degree of the defining polynomial.
- e_{HK} may not remain same under flat family of sheaves.

Harder-Narasimhan filtration

- X is a projective curve over a field K; V is a vector bundle on X.
- V is called semistable, if for any subbundle $V' \hookrightarrow V$, we have $\mu(V') \leq \mu(V)$.
- (Harder-Narasimhan '97) Let X be a smooth projective curve; Then $\exists!$ filtration by subbundles of V,

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V \tag{1.1}$$

  such that

  1. $V_1, V_2/V_1 \ldots, V/V_{l-1}$ are semistable vector bundles and
  2. $\mu(V_1) > \mu(V_2/V_1) > \cdots > \mu(V/V_{l-1})$.

Strong Harder-Narasimhan filtration

- char $K = p > 0$. V is strongly semistable if $F^{n*}V$ is semistable for all $n \geq 0$ where $F : X \longrightarrow X$ absolute Frobenius map.
- $K = \overline{K}$ and char $K = p > 0$.
- (Langer '04) Suppose X is smooth. Then $\exists\ n_0 > 0$ such that $F^{n*}V$ has a strong Harder-Narasimhan filtration, for all $n \geq n_0$.
- Let $0 = V_0 \subset V_1 \subset \cdots \subset V_l = F^{e*}V$ be a strong Harder-Narasimhan filtration of $F^{e*}V$. Set

$$a_i(V) := \frac{\mu(V_i/V_{i-1})}{\deg(\mathrm{X}).p^e} \text{ and } r_i(V) := \mathrm{rank}(V_i/V_{i-1}).$$

e_{HK} for projective curves

- $K = \overline{K}$ and char $K = p > 0$.
- R, a normal standard graded ring of dimension 2 over K and let $X = \text{Proj}(R)$.
- $I = (f_1, \ldots, f_r)$, homogeneous ideal of finite colength with $\deg(f_i) = d_i$.
- Consider the exact sequence

$$0 \longrightarrow V \longrightarrow \oplus_{i=1}^{r} \mathcal{O}_X(1 - d_i) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0.$$

- (Trivedi '05; Brenner '06)

$$e_{HK}(R, I) = \frac{\deg(X)}{2} \left(\sum_{i=1}^{l} r_i(V) a_i(V)^2 - \sum_{i=1}^{r} d_i^2 \right).$$

Hilbert-Kunz density function

- R, a standard graded Noetherian ring of dimension $d \geq 2$ over a perfect field K of char $p > 0$
- $I \subset R$ be a homogeneous ideal such that $\ell(R/I) < \infty$.
- M be a finitely generated non-negatively graded R-module.
- For $n \in \mathbb{N}$, $q = p^n$, define $f_n(M, I) : [0, \infty) \longrightarrow [0, \infty)$ as

$$\lambda \longmapsto \frac{1}{q^{d-1}} \ell\big((M/I^{[q]}M)_{\lfloor q\lambda \rfloor}\big)$$

- (Trivedi '18) $\{f_n(M, I)\}_n \longrightarrow f_{M,I}$ uniformly, where $f_{M,I} : [0, \infty) \longrightarrow [0, \infty)$ is a compactly supported continuous function such that

$$e_{HK}(M, I) = \int_0^\infty f_{M,I}(\lambda)\, d\lambda.$$

Properties of HK density function

[Trivedi '18]

- (Additive property) Let Λ be the set of minimal prime ideals P of R such that $\dim(R/P) = \dim(R)$. Then

$$f_{M,I} = \sum_{P\in\Lambda} f_{R/P,I+P/P}\ell_{R_P}(M_P).$$

- Suppose R is formally unmixed ring and $I \subseteq J$ are two homogeneous ideals of R. Then $f_{R,I} = f_{R,J}$ if and only if $J \subseteq I^*$.
- Let R and S standard graded rings, $I \subset R$ and $J \subset S$ be homogeneous ideals of finite colength.

  Then

$$F_{R\#S} - f_{R\#S,I\#J} = [F_R - f_{R,I}][F_S - f_{S,J}]$$

  where F_R, F_S and $F_{R\#S}$ are known polynomial functions, expressible using the Hilbert-Samuel multiplicity of the respective ring.

HK density function for curves

- $K = \overline{K}$ and char $K = p > 0$.
- $(R, \mathbf{m})$, normal standard graded domain of dimension 2.
- $X = \text{Proj } R$, $d = \deg(X)$.
- Let V be the syzygy bundle as before
- let $r_i = r_i(V)$ and let $a_i = a_i(V)$ be the normalised Harder-Narasimhan slopes, for $i = 1, \ldots, l$.
- Then

$$f_{R,\mathbf{m}}(\lambda) = \begin{cases} d(\lambda) & \text{if } 0 \leq \lambda < 1, \\ \sum_{j=i}^{l} -d(a_j r_j + r_j(\lambda - 1)) & \text{if } 1 - a_{i-1} \leq \lambda < 1 - a_i, \\ 0 & \text{if } \lambda \geq 1 - a_l. \end{cases}$$

- One can also describe $f_{R,I}$ where I is a homogeneous ideal of finite colength.

HK density function for curves

- $K = \overline{K}$ and char $K = p > 0$.
- $(R, \mathbf{m})$, normal standard graded domain of dimension 2.
- $X = \text{Proj } R$, $d = \deg(X)$.
- Let V be the syzygy bundle as before
- let $r_i = r_i(V)$ and let $a_i = a_i(V)$ be the normalised Harder-Narasimhan slopes, for $i = 1, \ldots, l$.
- Then

$$f_{R,\mathbf{m}}(\lambda) = \begin{cases} d(\lambda) & \text{if } 0 \leq \lambda < 1, \\ \sum_{j=i}^{l} -d(a_j r_j + r_j(\lambda - 1)) & \text{if } 1 - a_{i-1} \leq \lambda < 1 - a_i, \\ 0 & \text{if } \lambda \geq 1 - a_l. \end{cases}$$

- One can also describe $f_{R,I}$ where I is a homogeneous ideal of finite colength.

Maximum support of HK density function

- K perfect field of positive characteristic $p > 0$
- $(R, \mathbf{m})$ standard graded ring over K; $\dim R = d$
- I homogeneous ideal of finite colength, M a finitely generated R-module
- Let $\alpha_{M,I} = \text{Sup}\{\lambda \in [0, \infty) \mid f_{M,I}(\lambda) \neq 0\}$, i.e., $\alpha_{M,I}$ is the maximum support of the HK density function $f_{M,I}$.
- (Trivedi-Watanabe '20) If R is a domain, then $\alpha_{R,\mathbf{m}} = d$ if and only if R is regular.
- F-threshold of I with respect to J, (for $I \subseteq \sqrt{J}$) $c^J(I) = \lim_{n\to\infty} \frac{\max\{r \mid I^r \subseteq J^{[q]}\}}{q}$.
- (Trivedi-Watanabe '20) $\alpha_{R,I} \leq c^I(\mathbf{m})$.

Maximum support of HK density function

- K perfect field of positive characteristic $p > 0$
- $(R, \mathbf{m})$ standard graded ring over K; $\dim R = d$
- I homogeneous ideal of finite colength, M a finitely generated R-module
- Let $\alpha_{M,I} = \mathrm{Sup}\{\lambda \in [0, \infty) \mid f_{M,I}(\lambda) \neq 0\}$, i.e., $\alpha_{M,I}$ is the maximum support of the HK density function $f_{M,I}$.
- (Trivedi-Watanabe '20) If R is a domain, then $\alpha_{R,\mathbf{m}} = d$ if and only if R is regular.
- F-threshold of I with respect to J, (for $I \subseteq \sqrt{J}$) $c^J(I) = \lim_{n\to\infty} \frac{\max\{r \mid I^r \subseteq J^{[q]}\}}{q}$.
- (Trivedi-Watanabe '20) $\alpha_{R,I} \leq c^I(\mathbf{m})$.

Maximal support for projective curves

- (Trivedi-Watanabe '20) If R is strongly F-regular on the punctured spectrum Spec $R \setminus \{\mathbf{m}\}$ or
  R is a two dimensional domain and I is generated by homogeneous elements of the same degree, then $\alpha_{R,I} = c^I(\mathbf{m})$.
- $(R, \mathbf{m})$, normal standard graded domain of dimension 2.
- I is a homogeneous ideal generated in same degree.
- $X = \text{Proj } R$, $d = \deg(X)$.
- Let V be the syzygy bundle as before.
- (Trivedi-Watanabe '20) $c^I(\mathbf{m}) = 1 - a_{\min}(V)$.
- (Trivedi '20) I is any homogeneous ideal of finite colength. Then $\alpha_{R,I} = 1 - a_{min}(V_t)$ where V_t is the 'strong μ-reduction bundle' of V.

Projective toric variety

- $K = \overline{K}$ a field of characteristic $p > 0$.
- A toric pair (X, D):= X is a projective toric variety of dimension $d-1$ over K, with a very ample torus invariant Cartier divisor D.
- $P_D \subset \mathbb{Z}^{d-1} \otimes \mathbb{R} \simeq \mathbb{R}^{d-1}$, the associated very ample lattice polytope.
- $(R, \mathbf{m})$:= homogeneous coordinate ring of (X, D).
- $C_D = \text{Cone}(P_D \times 1) \subseteq (\mathbb{Z}^{d-1} \otimes \mathbb{R}) \times \mathbb{R}$.

Hilbert-Kunz multiplicity and density function of a toric pair

- $\mathcal{P}_D = \{p \in C_D \mid p \in (u, 1) + C_D, \text{ for every } u \in P_D \cap M\}$.
- (Eto 2000) $e_{HK}(R, \mathbf{m}) = \text{Vol}_d(\overline{\mathcal{P}_D})$.
- (M-Trivedi)

$$f_{R,\mathbf{m}}(\lambda) = \text{Vol}_{d-1}(\overline{\mathcal{P}_D} \cap \{z = \lambda\}) = \text{Vol}_{d-1}(\mathcal{P}_D \cap \{z = \lambda\})$$

  for all $\lambda \in [0, \infty)$.
- It is a piecewise polynomial function, independent of characteristic.

HK density function

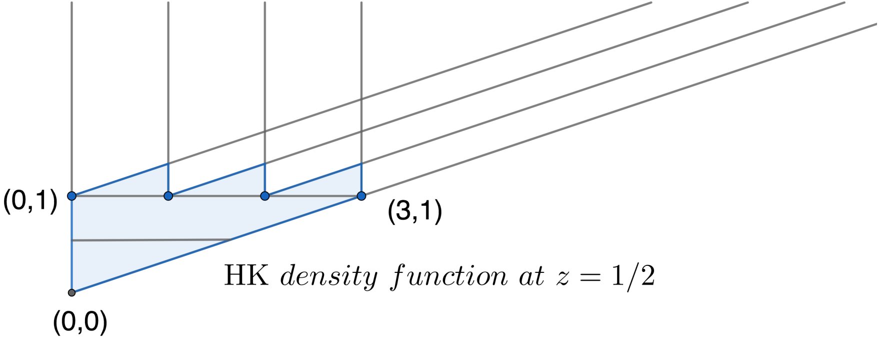

Figure: $f_{R,\mathbf{m}}$ for $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$

Example

- The toric pair $(X, D) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$.
- $(R, \mathbf{m})$, the associated coordinate ring with homogeneous maximal ideal $\mathbf{m}$.
- $\mathrm{HK}(R, \mathbf{m})(q) = q^3 + O(q)$.
- $$f_{R,\mathbf{m}}(\lambda) = \begin{cases} \frac{1}{2}\lambda^2 & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{2}\lambda^2 - \frac{3}{2}(\lambda-1)^2 & \text{if } 1 \leq \lambda < 2, \\ \frac{1}{2}\lambda^2 - \frac{3}{2}(\lambda-1)^2 + \frac{3}{2}(\lambda-2)^2 & \text{if } 1 \leq \lambda < 2. \end{cases}$$

β-density function (density function for the second coefficient of the Hilbert-Kunz function)

Question

1. *Does there exist a nice function* $\alpha_M : [0, \infty) \to \mathbb{R}$ *such that*
$$\int_0^\infty \alpha_M(x)\ dx = \tau(M, \mathbf{m}),$$
which may similarly refine the τ*-invariant in the graded case?*
2. *Does there exist a nice function* $g_{R,\mathbf{m}} : [0, \infty) \to \mathbb{R}$ *such that*
$$\int_0^\infty g_{R,\mathbf{m}}(x)\ dx = \beta(R, \mathbf{m}),$$
which may similarly refine the β*-invariant in the graded case?*

β-density function for $(R, \mathbf{m})$

Define $\{g_n(M) : [0, \infty) \longrightarrow \mathbb{R}\}_{n \in \mathbb{N}}$ given by

$$\lambda \mapsto \frac{1}{q^{d-2}} \left(\ell((M/\mathbf{m}^{[q]}M)_{\lfloor \lambda q \rfloor}) - \text{rank}(M) f_{R,\mathbf{m}}(\lfloor \lambda q \rfloor / q) q^{d-1} \right).$$

It follows g_n is a compactly supported function.

Theorem (M-Trivedi)

Outside a finite set $v(\mathcal{P}_D) \subseteq \mathbb{R}_{\geq 0}$, the sequence $\{g_n(R)\}_n$ converges uniformly to $g_{R,\mathbf{m}} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, given by

$$\lambda \mapsto \text{rVol}_{d-2}\left(\partial(\mathcal{P}_D) \cap \partial(C_D) \cap \{z = \lambda\}\right) - \frac{\text{rVol}_{d-2}\left(\partial(\mathcal{P}_D) \cap \{z = \lambda\}\right)}{2}.$$

$$\beta(R, \mathbf{m}) = \int_0^\infty g_{R,\mathbf{m}}(\lambda) d\lambda = \text{rVol}_{d-1}\left(\partial(\mathcal{P}_D) \cap \partial(C_D)\right) - \frac{\text{rVol}_{d-1}(\partial(\mathcal{P}_D))}{2}.$$

β-density function for $(R, \mathbf{m})$

Define $\{g_n(M) : [0, \infty) \longrightarrow \mathbb{R}\}_{n \in \mathbb{N}}$ given by

$$\lambda \mapsto \frac{1}{q^{d-2}} \left(\ell((M/\mathbf{m}^{[q]}M)_{\lfloor \lambda q \rfloor}) - \text{rank}(M) f_{R,\mathbf{m}}(\lfloor \lambda q \rfloor / q) q^{d-1} \right).$$

It follows g_n is a compactly supported function.

Theorem (M-Trivedi)

Outside a finite set $v(\mathcal{P}_D) \subseteq \mathbb{R}_{\geq 0}$, the sequence $\{g_n(R)\}_n$ converges uniformly to $g_{R,\mathbf{m}} : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$, given by

$$\lambda \mapsto \text{rVol}_{d-2}\left(\partial(\mathcal{P}_D) \cap \partial(C_D) \cap \{z = \lambda\}\right) - \frac{\text{rVol}_{d-2}\left(\partial(\mathcal{P}_D) \cap \{z = \lambda\}\right)}{2}.$$

$$\beta(R, \mathbf{m}) = \int_0^\infty g_{R,\mathbf{m}}(\lambda) d\lambda = \text{rVol}_{d-1}\left(\partial(\mathcal{P}_D) \cap \partial(C_D)\right) - \frac{\text{rVol}_{d-1}(\partial(\mathcal{P}_D))}{2}.$$

Example (continued)

$$g_n(R)(\lambda) = \begin{cases} \frac{1}{q}\left(\frac{3}{2}m+1\right) & \text{if } 0 \leq \lambda < 1, \\ \frac{1}{q}\left(-3m+\frac{9}{2}q-2\right) & \text{if } 1 \leq \lambda < 2, \\ \frac{1}{q}\left(\frac{3}{2}m-\frac{9}{2}q+1\right) & \text{if } 1 \leq \lambda < 2, \end{cases}$$

$$g_{R,\mathbf{m}}(\lambda) = \begin{cases} \frac{3}{2}\lambda & \text{if } 0 \leq \lambda < 1, \\ -3\lambda+\frac{9}{2} & \text{if } 1 \leq \lambda < 2, \\ \frac{3}{2}\lambda-\frac{9}{2} & \text{if } 2 \leq \lambda < 3, \end{cases}$$

and $\int_0^\infty g_{R,\mathbf{m}}(\lambda)\ d\lambda = 0$.

Monomial prime ideals of height one

- $\Gamma = \mathbb{N}\langle (P_D \cap M) \times \{1\} \rangle \subset \mathbb{R}^{d-1} \times \mathbb{R}$.
- There is a bijective correspondence

$$\{\text{monomial prime ideals of } R\} \leftrightarrow \{\text{faces of the cone } C_D\}$$

given by

$$C_F \leftrightarrow p_F := \text{ ideal generated by the set } \{\chi^u \mid \nu \in \Gamma \setminus C_F\}$$

where C_F is the face of the cone C_D corresponding to a face F of P_D.
- Under this correspondence

$$\{\text{height one monomial prime ideals of } R\} \leftrightarrow \{\text{facets of the cone } C_D\}$$

Class group of R

Suppose (X, D) is a projectively normal toric pair, i.e., R is an integrally closed domain.

- Then $\Gamma = C_D \cap \mathbb{Z}^d$.
- Suppose ht $p_F = 1$. The valuation v_{p_F} is the unique extension of the support form σ_F of C_D associated with the facet C_F.
- The divisorial monomial ideals of R are exactly the R-submodules of $R = K[\Gamma]$ whose monomial basis is determined by a system

$$\{x \in \mathbb{Z}^d \mid \sigma_F(x) \geq n_F, F \text{ is a facet of } P_D\}$$

  for $n_F \in \mathbb{Z}$.
- The class group of R is generated by the classes of the ideals p_F for where F runs over the set of facets of P_D.

Small density function

Let $\mathfrak{p} = p_F$ the height one monomial prime ideal of R corresponding to a face F of P_D. Define $\Psi_n : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\lambda \mapsto \frac{1}{q^{d-2}} \ell \left(\frac{m^{[q]} \cap \mathfrak{p}}{m^{[q]}\mathfrak{p}} \right)_{\lfloor q\lambda \rfloor}$$

- $H_{F,\mu} = \{x \in \mathbb{R}^d \mid \sigma_F(x) = \mu\}$ for all $\mu \in \mathbb{Q}_{\geq 0}$
- $\mu_{D,F} = \{\mu \in \mathbb{Q}_{\geq 0} \mid \sigma_F(u,1) = \mu$ for some $u \in (P_D \setminus F) \cap \mathbb{Z}^{d-1}\}$.

Definition

We set the map $\Psi_F : [0, \infty) \longrightarrow \mathbb{R}$ *by*

$$\Psi_F(\lambda) = \sum_{\mu \in \mu_{D,F}} \mathrm{rVol}_{d-2}(\partial(\mathcal{P}_D) \cap H_{F,\mu} \cap \{z = \lambda\}).$$

β-density function for the monomial prime ideals of height one

Theorem (M)

- *Outside a finite set $v_{D,F}$, the sequence of functions $\{\Psi_n\}$ converges uniformly to the functions Ψ_F. Moreover, Ψ_F is continuous on $\mathbb{R}_{\geq 0} \setminus v_{D,F}$.*
- *Set $\alpha_{\mathfrak{p}} = \Psi_F - f(R/\mathfrak{p}, \mathbf{m}/\mathfrak{p})$. Then $\int \alpha_{\mathfrak{p}}(\lambda) d\lambda = \tau(\mathfrak{p}, \mathbf{m})$.*

Extend to define a map $\alpha : \mathrm{Cl}(R) \longrightarrow \mathcal{L}^1([0,\infty))$ such that it is a group homomorphism.

Theorem (M)

Outside the finite set $v_{D,F}$, the sequence $\{g_n(\mathfrak{p})\}$ converges uniformly to the function $g_{\mathfrak{p},\mathbf{m}} = g_{R,\mathbf{m}} - \alpha_{\mathfrak{p}}$. Moreover, $\int g_{\mathfrak{p},\mathbf{m}} = \beta(\mathfrak{p}, \mathbf{m})$.

β-density function for product of two toric pairs

- $(R, \mathbf{m})$, $(S, \mathbf{n})$ homogeneous coordinate rings for the toric pairs (X, D) and (Y, D'), respectively.

- (M-Trivedi '20) Then,

  $$G_{R\#S} - g_{R\#S,\mathbf{m}\#\mathbf{n}} = (G_R - g_{R,\mathbf{m}})(F_S - f_{S,\mathbf{n}}) + (G_S - g_{S,\mathbf{n}})(F_R - f_{R,\mathbf{m}}).$$

  Here G_R and G_S are known polynomial functions involving second coefficient of the Hilbert-Samuel polynomial of $(R, \mathbf{m})$ and $(S, \mathbf{n})$, respectively.

- (M) $\alpha_{\mathfrak{p}\#S} = \alpha_{\mathfrak{p},\mathbf{m}}(F_S - f_{S,\mathbf{n}}) + (G_R - G_{\mathfrak{p}})f_{\mathfrak{p},\mathbf{m}}$.

Decomposition of $\mathcal{P}_D$ and $\overline{\mathcal{P}_D}$

We decompose $\overline{\mathcal{P}_D} = \cup_{j=1}^{s} P_j$ such that

- Each P_j is a d-dimensional convex rational polytope and $\overline{\mathcal{P}_D} = \cup_j P_j$.
- For $i \neq j$, $\dim(P_j \cap P_i) < d$, i.e., they do not intersect each other in the interior.
- $\dim(\partial(P_j) \cap \{z = \lambda\}) < d - 1$.
- For $\lambda \in [0, \infty)$, we have

$$f_{R,\mathbf{m}}(\lambda) = \mathrm{Vol}_{d-1}(\mathcal{P}_D \cap \{z = \lambda\})$$

$$= \mathrm{Vol}_{d-1}(\overline{\mathcal{P}_D} \cap \{z = \lambda\}) = \sum_j \mathrm{Vol}_{d-1}(P_j \cap \{z = \lambda\}).$$

Boundary of $\mathcal{P}_D$

Study the boundary of the Eto set $\mathcal{P}_D$ in terms of the facets of the rational polytopes P_i and the facets of the 'shifted cone' $(u, 1) + C_D$ where $u \in P_D \cap \mathbb{Z}^{d-1}$.

$g_n(R)(\lambda) = g_{R,\mathbf{m}}(\frac{\lfloor \lambda q \rfloor}{q}) + \frac{c_\lambda(n)}{q}$ such that $|c_\lambda(n)| < C$, a positive constant independent of λ and n.

Study the boundary of the set $\mathcal{P}_D$ 'parallel to the hyperplane $H_F = \{\sigma_F = 0\}$', i.e., one needs to understand the sets $\partial(\mathcal{P}_D) \cap H_{F,\mu}$.

$\Psi_n(\lambda) = \Psi_F(\frac{\lfloor \lambda q \rfloor}{q}) + \frac{c_\lambda(n)}{q}$ such that $|c_\lambda(n)| < C$, a positive constant independent of λ and n.

Bounding the 'correction' term

(McMullen) Let $P_1, \ldots, P_k \subset \mathbb{R}^d$ rational polytopes. Then $Q(P_1, \ldots, P_k; \mathbf{r}) = \#(\sum r_i P_i \cap \mathbb{Z}^d)$ where $\mathbf{r} = (r_1, \ldots, r_k) \in \mathbb{Q}^k_{\geq 0}$, is a rational quasipolynomial, i.e.,

$Q(P_1, \ldots, P_k; \mathbf{r}) = \sum_{|\mathbf{l}|<\mathbf{d}} p_{\mathbf{l}}(\mathbf{r})\mathbf{r}^{\mathbf{l}}$ where $\mathbf{l} = (l_1, \ldots, l_k) \in \mathbb{Z}_{\geq 0}$, $\mathbf{r}^{\mathbf{l}} = r_1^{l_1} \cdots r_k^{l_k}$, $|\mathbf{l}| = l_1 + \cdots + l_k$ and $p_{\mathbf{l}} : \mathbb{Q}^k \longrightarrow \mathbb{Q}$ is a periodic function.

(M-Trivedi) Extend a result of Henk and Linke about the 'polynomial' behaviour of the coefficient functions $p_{\mathbf{l}}(\mathbf{r})$.

Bounding the 'correction' term

Theorem (M-Trivedi)

Let $P \subseteq \mathbb{R}^{d-1} \times \mathbb{R} \simeq \mathbb{R}^d$ be a d-dimensional polytope. For $\lambda \in \mathbb{Q}_{\geq 0}$ such that $\lambda n \in \mathbb{Z}_{\geq 0}$, let $P_\lambda = P \cap \{z = \lambda\}$ and

$$\#(n(P_\lambda) \cap \mathbb{Z}^d) = \sum_{i=0}^{\dim(P_\lambda)} C_i(P_\lambda, n) n^i$$

be the Ehrhart quasipolynomial for P_λ. Then $C_i(P_\lambda, n) \leq c_i(P)$, a constant independent of λ and n and $C_{d-1}(P_\lambda) = \mathrm{rVol}_{d-2}(P_\lambda)$.

Watanabe's Example

- $$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$$
- $T = \mathbb{k}[x_1, \ldots, x_6]/I_2(A)$.
- (Huneke-McDermott-Monsky) $\beta(T, \mathbf{m}_T) = -1/4$.
- $(R, \mathbf{m}_R)$ and $(S, \mathbf{m}_S)$ are homogeneous coordinate ring of the pairs $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and $(\mathbb{P}^2, \mathcal{O}_{P^2}(1))$;
- $\{I_1, I_2\}$ and $\{J_1, J_2, J_3\}$ are height one monomial prime ideals of R and S respectively.

$$\alpha_{I_i \# S}(\lambda) = \begin{cases} -\frac{1}{2}\lambda^2 & \text{if } 0 \leq \lambda \leq 1, \\ -\frac{1}{2}\lambda^2 + 3(\lambda-1)^2 & \text{if } 1 \leq \lambda \leq 2, \\ -\frac{1}{2}\lambda^2 + 3\lambda - \frac{9}{2} & \text{if } 2 \leq \lambda \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

(continued)

$$\alpha_{R\#J_i}(\lambda) = \begin{cases} \lambda^2 & \text{if } 0 \leq \lambda \leq 1, \\ \lambda^2 + 6(\lambda - 1)^2 & \text{if } 1 \leq \lambda \leq 2, \\ -\lambda^2 + 3\lambda & \text{if } 2 \leq \lambda \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

- $\tau(I_i\#S) = -1/2$; $\tau(R\#J_j) = 1/2$.

References

- W. Bruns, J. Gubeladze, *Polytopes, rings, and K-theory*. Springer Monographs in Mathematics. Springer, Dordrecht, (2009).
- K. Eto, *Multiplicity and Hilbert–Kunz multiplicity of monoid rings*. Tokyo J. Math. 25 (2002), 241-245.
- W. Fullton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131, The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, (1993).
- C. Huneke, M. A. McDermott, P. Monsky, *Hilbert-Kunz functions for normal rings*. Math. Res. Lett. 11 (2004), 539-546.
- M. Mondal, V. Trivedi, *Hilbert-Kunz density function and and asymptotic Hilbert-Kunz multiplicity for projective toric varieties*. J. Algebra 520 (2019), 479-516.

References

M. Mondal, V. Trivedi, *Density function for the second coefficient of the Hilbert-Kunz function on projective toric varieties*. J. Algebraic Combin. 51 (2020), no. 3, 317–351.

M. Mondal, *β-density function for projective toric varieties*, J. Pure Appl. Algebra 226 (2022) Paper No. 106845, 21 pp.

P. Monsky, *The Hilbert-Kunz function.*, Math. Ann. 263 (1983), 43-49.

V. Trivedi, *Hilbert-Kunz density function and Hilbert-Kunz multiplicity*. Trans. Amer. Math. Soc. 370 (2018), 8403-8428.

Thank you