Seshadri constants of equivariant vector bundles on toric varieties

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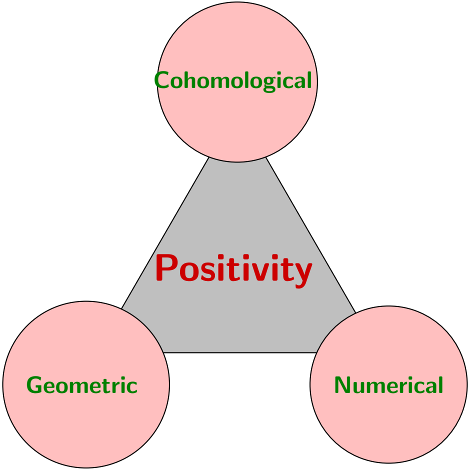
- "Positivity" of line bundles means that it has "many global sections".



Framework: All varieties are nonsingular projective defined over $\mathbb{C}$.

Let $\mathcal{L}$ be a line bundle on a variety X and $s_0, s_1, \ldots s_N$ be a $\mathbb{C}$-basis for $H^0(X, \mathcal{L})$. Then there is the associated **Kodaira map**

$$\phi_{\mathcal{L}} : X \setminus Bs(\mathcal{L}) \longrightarrow \mathbb{P}^N, \text{ defined by } x \longmapsto [s_0(x) : s_1(x) : \ldots : s_N(x)],$$

where $Bs(\mathcal{L}) := \mathbb{V}(s_0) \cap \ldots \cap \mathbb{V}(s_N)$ is the base locus of the line bundle $\mathcal{L}$.

- The line bundle $\mathcal{L}$ is called globally generated if $Bs(\mathcal{L}) = \emptyset$. In addition, if $\phi_{\mathcal{L}}$ defines a closed embedding $\phi_{\mathcal{L}} : X \hookrightarrow \mathbb{P}^N$, then $\mathcal{L}$ is said to be very ample.
- The line bundle $\mathcal{L}$ is called ample if there exists a positive integer m such that $\mathcal{L}^{\otimes m}$ is very ample.

Some criteria for ampleness

Theorem 1 (Nakai-Moishezon-Kleiman criterion)

Let $\mathcal{L}$ be a line bundle on a projective variety X. Then $\mathcal{L}$ is ample if and only if

$$\mathcal{L}^{\dim V} \cdot V > 0$$

for every positive dimensional irreducible subvariety $V \subseteq X$.

- A line bundle $\mathcal{L}$ is called numerically effective (nef) if $\mathcal{L} \cdot C \geq 0$ for all irreducible curves C in X.

Seshadri criterion for ampleness (1972)

A line bundle $\mathcal{L}$ on X is ample if and only if for every point $x \in X$ there exists a positive number ε such that $\frac{\mathcal{L} \cdot C}{\mathrm{mult}_x C} \geq \varepsilon$ for all curves C passing through x.

Lets look for optimal values of ε!

Definition 1 (Demailly(1992))

Let $\mathcal{L}$ be a nef line bundle on a complex projective variety X. For a point $x \in X$, the Seshadri constant of $\mathcal{L}$ at x is defined to be

$$\varepsilon(X, \mathcal{L}, x) := \inf_{x \in C} \frac{\mathcal{L} \cdot C}{\mathrm{mult}_x C} .$$

This numerical invariant measures the "local positivity" of the line bundle $\mathcal{L}$ at the point x.

Reformulation of Seshadri's ampleness criterion

A nef line bundle on X is ample if and only if $\varepsilon(\mathcal{L}) := \inf_{x \in X} \varepsilon(X, \mathcal{L}, x) > 0$.

What to look for

Guiding problems on Seshadri constants

- Computing Seshadri constants.
- Giving bounds on them.
- Checking if they are irrational (Nagata Conjecture -1958).

Let us look at some existing results and applications in this context:

- Let $\mathcal{L}$ be an **ample and globally generated** line bundle on a variety X, then $\varepsilon(X, \mathcal{L}, x) \geq 1$ for all $x \in X$.
- **Characterization of $\mathbb{P}^n$:**

  Bauer-Szemberg (2009): Let X be a smooth Fano variety of dimension n. Then $X = \mathbb{P}^n \Longleftrightarrow \varepsilon(X, -K_X, x) \geq n + 1$ for some $x \in X$.
- DiRocco (1999) has computed Seshadri constants of an ample line bundle over a toric variety at any fixed point.
- Ito (2014) has given bounds on Seshadri constants on an **arbitrary toric variety** at any point.

Seshadri constant for vector bundles

X : nonsingular complex projective variety, $\mathcal{E}$: vector bundle on X

$\pi : \mathbb{P}(\mathcal{E}) \to X$: projectivized bundle associated to $\mathcal{E}$

$\xi := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$: tautological line bundle on $\mathbb{P}(\mathcal{E})$

A vector bundle $\mathcal{E}$ on X is ample (resp. nef) if the tautological line bundle ξ is ample (resp. nef) on the projectivized bundle $\mathbb{P}(\mathcal{E})$.

Definition 2 (Hacon (2000), Fulger-Murayama (2021))

The Seshadri constant of a nef vector bundle $\mathcal{E}$ at $x \in X$ is defined to be

$$\varepsilon(X, \mathcal{E}, x) := \inf_{C \subset \mathbb{P}(\mathcal{E})} \frac{\xi \cdot C}{\mathrm{mult}_x \pi_* C},$$

where the infimum is taken over all curves C on $\mathbb{P}(\mathcal{E})$ that meet $\pi^{-1}(x)$ but not completely contained in $\pi^{-1}(x)$.

- Let $\mathcal{E}$ be an **ample and globally generated** vector bundle on a smooth complex projective curve X, then for all $x \in X$

$$\varepsilon(X, \mathcal{E}, x) \geq 1.$$

- **Another Characterization of $\mathbb{P}^n$:** Let X be a smooth Fano variety of dimension n with nef tangent bundle. Then

$$X = \mathbb{P}^n \iff \varepsilon(X, \mathscr{T}_X, x) > 0 \text{ for some } x \in X,$$

(Fulger-Murayama (2021)).

Some special cases

- Hacon (2000): Let $\mathcal{E}$ be a nef vector bundle on a smooth complex projective curve X, then for all $x \in X$

$$\varepsilon(X, \mathcal{E}, x) = \mu_{min}(\mathcal{E}),$$

where $\mu_{min}(\mathcal{E})$ denotes the smallest slope of any quotient bundle of $\mathcal{E}$. Here slope of the vector bundle $\mathcal{E}$ is $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}$.

- Fulger-Murayama (2021): If $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r$ is a nef vector bundle on a variety X, then for any $x \in X$

$$\varepsilon(X, \mathcal{E}, x) = \min_{1 \leq i \leq r} \{\varepsilon(X, \mathcal{E}_i, x)\}.$$

- Fulger-Murayama (2021): $\mathcal{E}$ **semistable discriminant zero** nef vector bundle of rank r on a variety X, then for all $x \in X$,

$$\varepsilon(X, \mathcal{E}, x) = \frac{1}{r} \varepsilon(X, \det(\mathcal{E}), x).$$

Complex Projective Space

- $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \mathbf{0}/\sim$, with homogeneous coordinates $[z_0 : \ldots : z_n]$.
- $T := \mathbb{P}^n \setminus \mathbb{V}(z_0 \cdots z_n) = \{[1 : t_1 \ldots : t_n] \in \mathbb{P}^n : t_i \neq 0\} \cong (\mathbb{C}^*)^n$, and T action on $\mathbb{P}^n$ given by:
  $(t_1, \ldots, t_n) \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : t_1 z_1 : \ldots : t_n z_n]$.
- $z, w \in \mathbb{P}^n$ are in the same orbit $\Longleftrightarrow$ they have the same support, where supp($z = [z_0 : \ldots : z_n]$)$= \{i : z_i = 0\}$.
- The torus T is the orbit of the point $[1 : \ldots : 1]$.
- There are $n+1$ fixed points $[1 : 0 : \ldots : 0], \ldots, [0 : 0 : \ldots : 1]$ and $n+1$ invariant hyperplane $(z_i = 0), i = 0, \ldots, n$.
- $\mathbb{P}^n$ is union of the $n+1$ invariant open sets $(z_i \neq 0), i = 0, \ldots, n$.

Definition 3

A toric variety X: A normal complex variety which contains a torus $T \cong (\mathbb{C}^*)^n$ as a dense open subset such that:

$$\begin{array}{ccc} T \times T & \longrightarrow & T \\ \big\downarrow & & \big\downarrow \\ T \times X & \longrightarrow & X \end{array}$$

Example 4

- $(\mathbb{C}^*)^n$, $\mathbb{C}^n$ and $\mathbb{P}^n$.

Theorem 2 (Fundamental theorem for toric varieties)

The category of toric varieties is equivalent to the category of fans.

$$X_\Delta \longleftrightarrow \Delta_X .$$

Combinatorial Description of Toric Variety

Fix a lattice $N \cong \mathbb{Z}^n$, $M = \text{Hom}(N, \mathbb{Z})$, $\langle \ , \ \rangle : M \times N \to \mathbb{Z}$.

$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, and $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$, the torus of N.

A strongly convex rational polyhedral cone (scrap) is a set

$\sigma = \text{Cone } (v_1, \ldots, v_s) = \{r_1 v_1 + \ldots + r_s v_s \in N_{\mathbb{R}} : r_i \in \mathbb{R}_{\geq 0}\}$, where

$v_1, \ldots, v_s$ in N and $\sigma \cap (-\sigma) = \{0\}$.

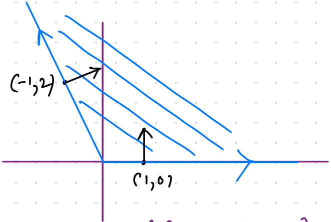
The dual cone is defined by

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

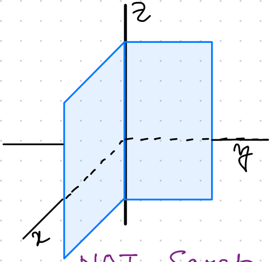
A face τ of σ is $\tau := \sigma \cap u^{\perp}$ for some $u \in \sigma^{\vee}$.

$S_{\sigma} = \sigma^{\vee} \cap M$ finitely generated semigroup $\rightsquigarrow \mathbb{C}[S_{\sigma}] = \mathbb{C}[\chi^m : m \in S_{\sigma}]$ finitely generated $\mathbb{C}$-algebra.

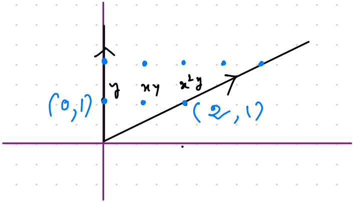
$U_{\sigma} := \text{Spec}(\mathbb{C}[S_{\sigma}])$ affine toric variety corresponding to σ.



$\sigma = \text{Cone}((1,0), (-1,2))$



NOT Scrap



$\sigma^{\vee} = \text{Cone}((2,1), (0,1))$

$$U_\sigma = \text{Spec}\, \mathbb{C}[y, xy, x^2y]$$

$$= \text{Spec}\, \frac{\mathbb{C}[x_1, x_2, x_3]}{(x_1x_3 - x_2^2)}$$

Fan

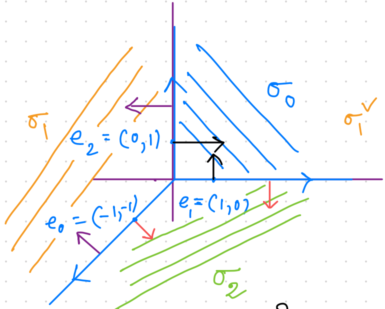
Definition 5

Fan Δ: a finite collection of scrap cones in $N_{\mathbb{R}}$ satisfying:

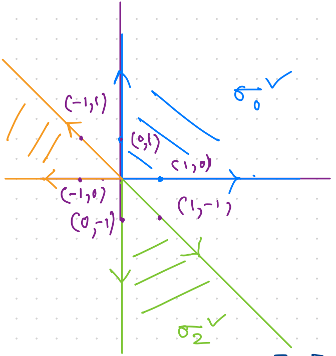
(1) If $\sigma \in \Delta$ and τ is a face of σ, then $\tau \in \Delta$.

(2) If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of each.

- Let $\tau \preceq \sigma$, then $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m)$, where $m \in \sigma^\vee \cap \tau^\perp \cap M$.
  $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$ and $U_\tau = (U_\sigma)_{\chi^m}$
  For the face $\{0\}$ of σ, we see that U_σ contains $U_0 = \text{Spec}(\mathbb{C}[M]) = T$ as a dense open subset.
- The toric variety arising from the fan Δ is denoted by $X(\Delta)$.



Fan Δ in $\mathbb{R}^2$



$$U_{\sigma_0} = \operatorname{Spec} \mathbb{C}[x,y] \simeq \operatorname{Spec} \mathbb{C}\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}\right]$$

$$U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[yx^{-1}, x^{-1}] \simeq \operatorname{Spec} \mathbb{C}\left[\frac{z_0}{z_1}, \frac{z_2}{z_1}\right]$$

$$U_{\sigma_2} = \operatorname{Spec} \mathbb{C}[xy^{-1}, y^{-1}] \simeq \operatorname{Spec} \mathbb{C}\left[\frac{z_0}{z_2}, \frac{z_1}{z_2}\right]$$

$$X(\Delta) = \mathbb{P}^2 \; ; \quad U_{\sigma_i} = U_i \quad i = 0,1,2.$$

- Let $X(\Delta)$ be a nonsingular toric variety of dimension n.
- Distinguished point $x_\sigma \in U_\sigma \rightsquigarrow$ maximal ideal $\langle (\chi^u - 1), \chi^v : u \in S_\sigma \cap \sigma^\perp, v \notin \sigma^\perp \rangle \subset \mathbb{C}[S_\sigma]$.
- The point x_σ is fixed under T-action $\Leftrightarrow$ dim $\sigma = n$.
- Denote by $O(\sigma) := T \cdot x_\sigma$. Then $U_\sigma = \bigsqcup_{\tau \preceq \sigma} O(\tau)$.
- The orbit closure $V(\tau) := \overline{O(\tau)} = \bigsqcup_{\tau \preceq \sigma} O(\sigma)$ toric variety of dimension $n -$ dim σ and any irreducible closed T-invariant subvarieties of $X(\Delta)$ are of this form.
- For $\rho \in \Delta(1)$, $D_\rho = V(\rho)$ is a T-invariant prime divisor. We have the exact sequence

$$M \xrightarrow{m \mapsto \text{div}(\chi^m)} \bigoplus_{\rho \in \Delta(1)} \mathbb{Z} D_\rho \longrightarrow \text{Pic}(X(\Delta)) \to 0.$$

Combinatorics of toric varieties: Summary

- Cone $\sigma \in \Delta \rightsquigarrow$ affine variety U_σ, distinguished point $x_\sigma \in U_\sigma$.
- x_σ is a torus fixed point $\Leftrightarrow \sigma \in \Delta$ is n-dimensional.
- 1-dimensional cone $\rho \in \Delta$ $\rightsquigarrow$ invariant divisors D_ρ.
- $(n-1)$-dimensional cone $\tau \in \Delta$ $\rightsquigarrow$ invariant curves $V(\tau) \cong \mathbb{P}^1$.

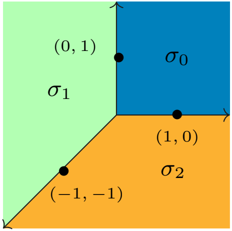


Figure: Fan of $\mathbb{P}^2$

Toric vector bundle

A T-equivariant vector bundle or toric vector bundle: A vector bundle $\pi : \mathcal{E} \to X$ on X with a lift of the action of T on the total space $\mathcal{E}$ in such a way that:

1. the projection map π is equivariant, i.e., for all $e \in \mathcal{E}$ and $t \in T$ the following diagram commutes:

$$
\begin{array}{ccc}
T \times \mathcal{E} & \longrightarrow & \mathcal{E} \\
(t, e) & \longmapsto & t \cdot e \\
{\scriptstyle Id \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
(t, \pi(e)) & \longmapsto & t \cdot \pi(e) \\
T \times X & \longrightarrow & X
\end{array}
$$

2. the torus T acts linearly on the fibers.

Example 6

line bundle, tangent bundle, cotangent bundle

Criterion for existence of Toric structure

- A vector bundle $\mathcal{E}$ can be endowed with an equivariant structure if and only if

$$t^*\mathcal{E} \cong \mathcal{E} \text{ for all } t \in T,$$

  where $t^*\mathcal{E}$ is the pull back of $\mathcal{E}$ via the map $X \xrightarrow{t\cdot} X$. [Klyachko (1990)]
- If indecomposable toric bundles $\mathcal{E}$ and $\mathcal{F}$ are isomorphic as ordinary bundles, then $\mathcal{E}$ is equivariantly isomorphic to $\mathcal{F} \otimes \mathrm{div}(\chi^m)$ for some character $m \in M$.

  Toric structure on a vector bundle $\mathcal{E}$ is not unique.

Klyachko(1990)

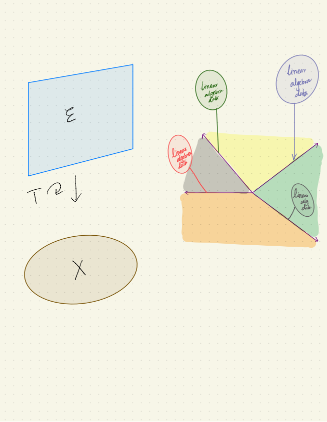
$\mathcal{E}$: rank r toric vector bundle on X,

$E = \mathcal{E}(1_T)$: the fiber at $1_T \in T \subset X$.

$$\mathcal{E} \longleftrightarrow E \supset \ldots \supset E^\rho(i) \supset E^\rho(i+1) \supset \ldots \mathbf{0},$$

rays of Δ satisfying compatibility condition: for any $\sigma \in \Delta$, there exists a $\widehat{T}_\sigma$-grading $E = \bigoplus_{\chi \in \widehat{T}_\sigma} E^{[\sigma]}(\chi)$, such that

$$E^\rho(i) = \bigoplus_{\langle \chi, v_\rho \rangle \geqslant i} E^{[\sigma]}(\chi) \text{ for all } \rho \in \sigma(1).$$



Example 7 (Filtrations for line bundle)

Let $\mathcal{L} = \mathcal{O}_X(D)$ *for some* T*-invariant divisor* $D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$. *Let* $\{m_\sigma\}_{\sigma \in \Delta}$ *be the Cartier data for* D, *i.e.* $D|_{U_\sigma} = \text{div}(\chi^{-m_\sigma})|_{U_\sigma}$ *and* $\langle m_\sigma, v_\rho \rangle = -a_\rho$. *Hence* $\mathcal{L} = \sqcup_{\sigma \in \Delta} (U_\sigma \times \mathbb{C}) \, / \sim$ *where*

$$U_{\sigma \cap \tau} \times \mathbb{C} \xrightarrow{\cong} U_{\sigma \cap \tau} \times \mathbb{C} \text{ given by}$$

$$(x, c) \longmapsto (x, \chi^{m_\sigma - m_\tau}(x)c).$$

The action of T *on* $U_\sigma \times \mathbb{C}$ *is given by* $t \cdot (x, c) = (t \cdot x, \chi^{-m_\sigma}(t)c)$, *where* $t \in T$ *and* $(x, c) \in U_\sigma \times \mathbb{C}$.

$T_\sigma :=$ *Stab*(x_σ) *acts on the fiber over* x_σ *by* χ^{-m_σ}. *There exists a* $\widehat{T}_\sigma$*-grading* $L = L^{[\sigma]}(\chi^{-m_\sigma})$. *Hence*

$$L^\rho(i) = \bigoplus_{\langle \psi, v_\rho \rangle \geqslant i} L^{[\sigma]}(\psi) = \begin{cases} \mathbb{C} & i \leqslant \langle -m_\sigma, v_\rho \rangle = a_\rho \\ 0 & i > \langle -m_\sigma, v_\rho \rangle = a_\rho \end{cases} .$$

Example 8 (Filtrations for tangent bundle and cotangent bundle)

The filtrations $\left(\mathscr{T}, \{\mathscr{T}^{\rho}(i)\}_{\rho\in\Delta(1), i\in\mathbb{Z}}\right)$ associated to $\mathcal{T}_X$ are given by

$$\mathscr{T}^{\rho}(i) = \begin{cases} N_{\mathbb{C}} & i \leqslant 0 \\ \text{Span } (v_\rho) & i = 1 \\ 0 & i > 1 \end{cases}$$

The filtrations $\left(\Omega, \{\Omega^{\rho}(i)\}_{\rho\in\Delta(1), i\in\mathbb{Z}}\right)$ for cotangent bundle Ω_X are given by:

$$\Omega^{\rho}(i) = \begin{cases} M_{\mathbb{C}} & i < 0 \\ \text{Span } (v_\rho)^{\perp} & i = 0 \\ 0 & i \geq 1, \end{cases}$$

for any ray $\rho \in \Delta(1)$.

Restriction to invariant curves

For each cone $\sigma \in \Delta$, there is a decomposition into eigenspaces as follows

$$E = \bigoplus_{u \in \mathbf{u}(\sigma)} L_u^\sigma,$$

where $\mathbf{u}(\sigma) \subset M$ is the associated characters of the toric vector bundle $\mathcal{E}$.

(Hering-Mustață-Payne (2010)): $\{\mathbf{u}(\sigma)\}_\sigma$ determines the restriction of $\mathcal{E}$ to the invariant curves.

Ample and nef toric vector bundles

(Hering-Mustață-Payne (2010)): A toric vector bundle on a complete toric variety is nef or ample if and only if its restriction to every invariant curve is nef or ample, respectively.

Seshadri constant for toric vector bundle

- X toric variety; $x \in X$ a **torus fixed point** and $\mathcal{E}$ a nef toric vector bundle on X. Then

$$\varepsilon(X, \mathcal{E}, x) = \min\{\mu_{min}(\mathcal{E}|_C) \mid x \in C \text{ and } C \text{ is an invariant curve}\}$$

(Hering-Mustață-Payne (2010)).

Goal: To compute Seshadri constant at arbitrary points.

Recall: to compute Seshadri constant at $x \in X$, we need to compute the ratios

$$\frac{\xi \cdot C}{\mathrm{mult}_x \pi_* C}, \quad \text{for all } C \subset \mathbb{P}(\mathcal{E}).$$

Key ingredient: the description of the Mori cone $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E}))$: the closed cone of curves of the projectived bundle $\mathbb{P}(\mathcal{E})$.

X: toric variety; $\mathcal{E}$: toric vector bundle on X; $l_1, \ldots, l_m$: invariant curves in X.

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}|_{l_i}) & \overset{\eta_i}{\hookrightarrow} & \mathbb{P}(\mathcal{E}) \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi} \\ l_i & \hookrightarrow & X \end{array}$$

- Since $\mathbb{P}(\mathcal{E}|_{l_i})$ is a toric variety, there is an invariant fiber curve Σ_i and invariant section curve Ω_i such that $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E}|_{l_i}) = \text{Cone}(\Sigma_i, \Omega_i)$.

Proposition 9 (Hering-Mustață-Payne (2010))

Take $C_0 := \eta_i(\Sigma_i)$ and $C_i := \eta_i(\Omega_i)$, then the Mori cone is given by

$$\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E})) = \Big\{ a_0 C_0 + \cdots + a_m C_m \mid a_i \in \mathbb{R}_{\geq 0} \text{ for } i = 0, \ldots, m \Big\}.$$

In particular, $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{E})$ is a polyhedral cone.

Theorem 10 (__ - Khan - Aditya)

*Let $\mathcal{E}$ be a "**nice**" nef equivariant vector bundle of rank r on the projective space $X = \mathbb{P}^n$ $(n \geq 2)$. Then for any point $x \in X$, we have*

$$\varepsilon(\mathcal{E}, x) = \min_{1 \leq i \leq m} \{\mu_{min}(\mathcal{E}|_{l_i})\}.$$

Example 11

- Uniform bundle: a bundle of splitting type $(a_1, \ldots, a_r)$, i.e., for any line $l \subset \mathbb{P}^n$, we have

$$\mathcal{E}|_l \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r).$$

  - $\mathscr{T}_{\mathbb{P}^n}$ is a uniform bundle with splitting type (2,1, …,1), hence for any $x \in \mathbb{P}^n$ the Seshadri constant is given by

$$\varepsilon(\mathscr{T}_{\mathbb{P}^n}, x) = 1.$$

Hirzebruch surface

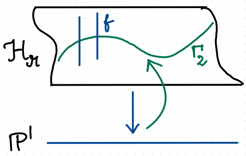


Figure: $\mathcal{H}_{c_{1,2}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(c_{1,2}))$

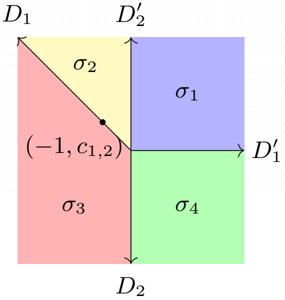


Figure: Fan for $\mathcal{H}_{c_{1,2}}$

- We have $D_1 \equiv D_1' \equiv f$,

  $D_2' \equiv D_2 - c_{1,2} D_1 \equiv \Gamma_2$
- The Picard group is

  $\mathrm{Pic}(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2$.
- The Nef cone is

  $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}D_1 \oplus \mathbb{R}_{\geq 0}D_2$,

  assuming $c_{1,2} \geq 0$.
- The Mori cone is

  $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}\Gamma_2 \oplus \mathbb{R}_{\geq 0}f$.

Theorem 12 (__ - Khan- Aditya)

Let $\mathcal{E}$ be an equivariant nef vector bundle of rank r on the Hirzebruch surface $X_2 = \mathcal{H}_{c_{1,2}}$ satisfying the following conditions:

$$\mu_{min}(\mathcal{E}|_{D_1}) = \mu_{min}(\mathcal{E}|_{D'_1}) \text{ and } \mu_{min}(\mathcal{E}|_{D_2}) \geq \mu_{min}(\mathcal{E}|_{D_1}).$$

Then for any $x \in X_2$, the Seshadri constant is given by:

$$\varepsilon(X_2, \mathcal{E}, x) = \begin{cases} min\{\mu_{min}(\mathcal{E}|_{D_1}), \mu_{min}(\mathcal{E}|_{D'_2})\}, & \text{if } x \in \Gamma_2, \\ \mu_{min}(\mathcal{E}|_{D_1}), & \text{if } x \notin \Gamma_2. \end{cases}$$

Seshadri constants of line bundles on Hirzebruch surfaces have been computed by Syzdek (2005), García (2006), Hanumanthu-Mukhopadhyay (2017).

Example 13

Consider the tangent bundle $\mathcal{E} = \mathscr{T}_{X_2}$ on the Hirzebruch surface X_2. Then the associated filtrations $(E, \{E^i(j)\}_{i=1,\ldots,4;\, j\in\mathbb{Z}})$ are given by:

$$E^i(j) = \begin{cases} \mathbb{C}^2 & j \leqslant 0 \\ \text{Span } (v_i) & j = 1 \\ 0 & j > 1 \end{cases}.$$

$\mathcal{E} \otimes \mathcal{O}(D)$ is nef, where $D = a_1 D_1 + a_2 D_2$, $a_1 \geq c_{1,2}$, $a_2 \geq 0$.

$(\mathcal{E} \otimes \mathcal{O}(D))|_{D_1'} = \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_2), (\mathcal{E} \otimes \mathcal{O}(D))|_{D_1} = \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_2),$

$(\mathcal{E} \otimes \mathcal{O}(D))|_{D_2'} = \mathcal{O}_{\mathbb{P}^1}(a_1 - c_{1,2}) \oplus \mathcal{O}_{\mathbb{P}^1}(2 + a_1),$

$(\mathcal{E} \otimes \mathcal{O}(D))|_{D_2} = \mathcal{O}_{\mathbb{P}^1}(a_1 + c_{1,2}\, a_2 + c_{1,2}) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1 + c_{1,2}\, a_2 + 2).$

The Seshadri constant is given by

$$\varepsilon(\mathcal{E} \otimes \mathcal{O}(D), x) = \begin{cases} \min\{a_1 - c_{1,2},\, a_2\}, & \text{if } x \in \Gamma_2, \\ a_2, & \text{if } x \notin \Gamma_2. \end{cases}$$

Bott towers

Bott towers are a particular class of nonsingular projective toric varieties. They were constructed by Grossberg-Karshon (1994).

For an integer $n \geq 0$, a **Bott tower of height** n

$$X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = \{\text{point}\}$$

is defined inductively as an iterated $\mathbb{P}^1$–bundle so that

$$X_k = \mathbb{P}(\mathcal{O}_{X_{k-1}} \oplus \mathcal{L}_{k-1})$$

for a line bundle $\mathcal{L}_{k-1}$ over X_{k-1}.

So $X_1 = \mathbb{P}^1$ and X_2 is a Hirzebruch surface and so on.

Fan structure of a Bott tower

- Let $T \cong (\mathbb{C}^*)^n$ be an algebraic torus with character lattice $M := \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^n$ and the cocharacter lattice $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.
- Let Δ_n be an n-dimensional nonsingular complete fan in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ which defines the toric variety X_n under the action of the torus T. the edges are

$$\begin{aligned} v_1 &= e_1, \cdots, v_n = e_n, \\ v_{n+1} &= -e_1 + c_{1,2}e_2 + \ldots + c_{1,n}e_n, \\ &\vdots \\ v_{n+i} &= -e_i + c_{i,i+1}e_{i+1} + \ldots + c_{i,n}e_n, 1 \leq i < n, \\ v_{2n} &= -e_n. \end{aligned} \tag{0.1}$$

- The maximal cones are generated by these edges such that no cone contains both the edges v_i and v_{n+i} for $i = 1, \cdots, n$.

Picard group of a Bott tower

- It follows that any k-th stage Bott tower arises from a collection of integers $\{c_{i,j}\}_{1\leq i<j\leq n}$ as in (0.1). These integers are called the *Bott numbers* of the given Bott tower.
- We will restrict our attention to the case when the Bott numbers $\{c_{i,j}\}_{\{1\leq i<j\leq n\}}$ are **all positive integers**.
- The Picard group of the Bott tower is

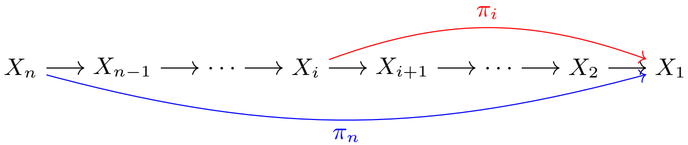
$$\text{Pic}(X_n) = \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_n \,,$$

where D_i denote the invariant prime divisor corresponding to the edge v_{n+i}.

Theorem 14 (Khan, _ (2019))

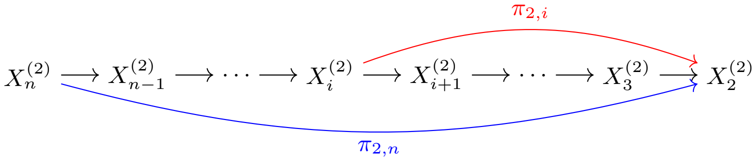
Let $D = \sum_{i=1}^{k} a_i D_i$ be a Cartier divisor on X_n. Then D is ample (respectively, nef) if and only if $a_i > 0$ (respectively, $a_i \geq 0$) for all $i = 1, \cdots, n$.

Fix a point $x \in X_n$. Set $X_i^{(1)} := X_i$ for every $1 \leq i \leq n$. For every $2 \leq i \leq n$, consider



Define $X_i^{(2)} := \pi_i^{-1}(\pi_n(x))$ for $i = 2, \ldots, n$. Note that $x \in X_n^{(2)}$.

Then



is a Bott tower.

For every $3 \leq i \leq n$, define $X_i^{(3)} := \pi_{2,i}^{-1}(\pi_{2,n}(x))$.

$$
\begin{array}{ccccccccccc}
\mathbb{P}^1 = X_n^{(n)} & \subset & X_n^{(n-1)} & \subset & \cdots & \subset & X_n^{(3)} & \subset & X_n^{(2)} & \subset & X_n^{(1)} \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{P}^1 = X_{n-1}^{(n-1)} & \subset & \cdots\cdot & \subset & X_{n-1}^{(3)} & \subset & X_{n-1}^{(2)} & \subset & X_{n-1}^{(1)} \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \cdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \mathbb{P}^1 = X_3^{(3)} & \subset & X_3^{(2)} & \subset & X_3^{(1)} \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & \mathbb{P}^1 = X_2^{(2)} & \subset & X_2^{(1)} \\
 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & \mathbb{P}^1 = X_1^{(1)}
\end{array}
$$

Proposition 15 (Biswas-__-Hanumanthu-Khan)

Each vertical tower is a Bott tower with positive Bott numbers.

Let us consider the composition of section maps

$$X_n^{(i)} \longrightarrow X_{n-1}^{(i)} \longrightarrow \cdots \longrightarrow X_{i+1}^{(i)} \longrightarrow X_i^{(i)}, \qquad \sigma_i : X_i^{(i)} \to X_n^{(i)}$$

for $1 \leq i \leq n$.

- Define $\Gamma_n^{(i)} := \sigma_i(X_i^{(i)}) \subset X_n^{(i)}$.
- We have $\Gamma_n^{(i)} \subset X_n^{(i)}$ for each i and $\Gamma_n^{(n)} = X_n^{(n)}$.
- We denote $\Gamma_n^{(1)}$ also by Γ_n.

Proposition 16 (Biswas-__-Hanumanthu-Khan)

The curves $\Gamma_n, \Gamma_n^{(2)}, \cdots, \Gamma_n^{(n)}$ span $\overline{\mathrm{NE}}(X_n)$, and they are dual to $D_1, \cdots, D_n$.

Theorem 17 (Biswas-__-Hanumanthu-Khan)

The Seshadri constant of a nef line bundle $\mathcal{L}$ *on* X_n *at a point* x *is given as follows:*

$$\varepsilon(X_n,\, \mathcal{L},\, x) \;=\; \min_{\mathrm{i}} \left\{ \mathcal{L} \cdot \Gamma_n^{(i)} \;\middle|\; x \in \Gamma_n^{(i)} \right\}.$$

Corollary 18

Let $\mathcal{L} \equiv a_1 D_1 + \ldots + a_n D_n$ *be a nef line bundle on* X_n.

- $\varepsilon(\mathcal{L}, 1) \;=\; a_n$.
- $\varepsilon(\mathcal{L}) \;=\; \mathit{min}\; \{a_1, \ldots, a_n\}$.

Theorem 19 (__ - Khan- Aditya)

Let $\mathcal{E}$ *be an equivariant nef vector bundle of rank* r *on* X_3 *satisfying "certain" conditions. Then the Seshadri constants of* $\mathcal{E}$ *at any* $x \in X_3$ *are given by*

$$\varepsilon(X_3, \mathcal{E}, x) = \min_{\mathrm{i}} \left\{ \mu_{min}(\mathcal{E}|_{\Gamma_3^{(i)}}) \;\middle|\; x \in \Gamma_3^{(i)} \right\}.$$

Example 20

Let $\mathcal{L} \equiv D_1 + 3D_2 + 8D_3 + 4D_4 \in \mathrm{Pic}(X_4)$ and $x \in X_4$. Then

$$\varepsilon(X_4, \mathcal{L}, x) = \begin{cases} 1, & \text{if } x \in \Gamma_4, \\ 3, & \text{if } x \notin \Gamma_4, x \in \Gamma_4^{(2)}, \\ 4, & \text{if } x \notin \Gamma_4, x \notin \Gamma_4^{(2)}. \end{cases}$$

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Thank You