Hilbert-Kunz multiplicity of powers of an ideal

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Outline of the talk

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Reductions and Integral closure of ideals

1. **Definition.** **(Northcott-Rees, 1954).** Let I and J be ideals of a commutative Noetherian ring. If $J \subseteq I$ and $JI^n = I^{n+1}$ for some n then J is called a **reduction of I** . The smallest reduction of I is called a **minimal reduction of I.**
2. The **reduction number** of I wrt J is the number $r_J(I) = \min\{n \mid JI^n = I^{n+1}\}$.
3. **Definition.** Let R be a commutative ring, I an ideal of R. An element $a \in R$ is called **integral over I,** if there exist $a_i \in I^i$ for $i = 1, 2, \ldots, n$ so that a satisfies:

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

4. The integral closure of $I := \bar{I} = \{a \in R \mid a \text{ is integral over } I\} \subseteq \sqrt{I}$.
5. An ideal I is called complete if $\bar{I} = I$.
6. **Proposition.** **(Northcott-Rees)** Let R be a ring and $J \subset I$ be ideals of R. Then J is a reduction of $I \iff I$ is integral over J.
7. **Theorem.** **(O. Zariski, 1938)** Product of complete ideals is complete in $k[x, y]$ where k is an algebraically closed field of characteristic zero.

The Hilbert-Samuel multiplicity of an ideal

1. Recall that if I is an $\mathfrak{m}$-primary ideal of a d-dimensional local ring $(R, \mathfrak{m})$ then the **Hilbert-Samuel function** of I is defined as $H_I(n) = \ell(R/I^n)$.
2. For large n, $H_I(n)$ is given by a polynomial $P_I(x)$ called the **Hilbert polynomial** of I. We write it as

$$P_I(x) = e(I)\binom{x+d-1}{d} - e_1(I)\binom{x+d-2}{d-1} + \cdots + (-1)^n e_d(I).$$

3. The coefficient $e(I), e_1(I), \ldots, e_d(I)$ are integers.
4. The coefficient $e(I)$ is called the **multiplicity of I.**
5. **Definition:** A local ring R is called **unmixed** if for every associated prime $\mathfrak{p}$ of the $\mathfrak{m}$-adic completion $\widehat{R}$ we have $\dim \widehat{R}/\mathfrak{p} = \dim R$.
6. **Theorem:** **(Northcott-Rees, 1954, 1961)** If J is a reduction of I, then $e(I) = e(J)$. The converse is true if R is an unmixed local ring.

Nagata's Theorem about regular local rings

1. Suppose that $(R, \mathfrak{m})$ is a d-dimensional regular local ring. Then
$$G(\mathfrak{m}) = \bigoplus_{n=0}^{\infty} \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \simeq k[X_1, X_2, \ldots, X_d].$$
2. Therefore $\ell(R/\mathfrak{m}^n) = \binom{n+d-1}{d}$ for all $n \geq 0$. Hence $e(\mathfrak{m}) = 1$.
3. The converse is not true. Consider $R = k[[X, Y, Z]]/(ZX, ZY)$.
4. Then $e(\mathfrak{m}) = 1$ but R is not regular since $\dim R = 2$ and $\mu(\mathfrak{m}) = 3$.
5. **Theorem.** **(M. Nagata, 1960)** A local ring $(R, \mathfrak{m})$ is regular if and only if R is unmixed and $e(\mathfrak{m}) = 1$.

Regular local rings and the Hilbert-Kunz function

1. **Definition.** Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p.
2. Let $q = p^e$. For an ideal $J = (a_1, a_2, \ldots, a_n)$, the q^{th}-Frobenius power of J is defined as $J^{[q]} = (a_1^q, a_2^q, \ldots, a_n^q)$.
3. The **Hilbert-Kunz function** of an $\mathfrak{m}$-primary ideal I is defined as
$$HK_I(q) = \ell(R/I^{[q]}).$$
4. Kunz considered $\lim_{q\to\infty} \frac{\ell(R/I^{[q]})}{q^d}$ and mentioned that it may not exist.
5. But Paul Monsky (1983) proved that it does exist and named it as the Hilbert-Kunz multiplicity of I. Therefore $e_{HK}(I) = \lim_{q\to\infty} \frac{\ell(R/I^{[q]})}{q^d}$.
6. **Theorem. (E. Kunz, 1969) :** Suppose $\dim R = d$. Then $\ell(R/\mathfrak{m}^{[q]}) \geq q^d$. The equality holds for some q if and only if R is regular.
7. When R is regular local then $\ell(R/\mathfrak{m}^{[q]}) = q^d$ for all q.
8. Therefore If R is a regular local ring then $e_{HK}(\mathfrak{m}) = 1$.
9. **Theorem.** Let R be a regular local ring of prime characteristic p. Then for any $\mathfrak{m}$-primary ideal I, $e_{HK}(I) = \ell(R/I)$.

Hilbert-Kunz multiplicity and tight closure

1. **Theorem.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then
$$\frac{e(I)}{d!} \leq e_{HK} \leq e(I).$$
2. **Corollary.** If $\dim R = 1$ then $e(I) = e_{HK}(I)$.
3. **Theorem.** **(Watanabe-Yoshida 2000)** An unmixed local ring $(R, \mathfrak{m})$ is regular if and only if $e_{HK}(\mathfrak{m}) = 1$.
4. Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ be the minimal primes of R and $R^\circ = R \setminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r)$. For an ideal I, the tight closure of I is the ideal
$$I^* = \{x \in R \mid \text{ there exists } c \in R^\circ \text{ such that } cx^q \in I^{[q]} \text{ for all large } q.\}$$
5. **Theorem.** **(Hochster-Huneke,1987)** (a) If R is a regular ring then all ideals I, $I = I^*$. (b) Let $J \subseteq I$ be $\mathfrak{m}$-primary ideals with $I^* = J^*$, then $e_{HK}(I) = e_{HK}(J)$. (c) If R is an unmixed local ring, then the converse also holds.
6. **Definition.** R is called **weakly F-regular** if $I = I^*$ for all R-ideals I.

Structure of local rings via the HK multiplicity

1. **Watanabe-Yoshida, 2001.** Let R be CM and $e(R) = 2$. Then R is F-rational if and only if $e_{HK}(R) < 2$.
2. Let $(R, \mathfrak{m}, k)$ be CM of dimension 2. Then $e_{HK}(R) \geq (e(R) + 1)/2$.
3. If $k = \overline{k}$ then $e_{HK}(R) = (e(R) + 1)/2 \iff gr_{\mathfrak{m}}(R) \simeq k[x, y]^{(e(R))}$.
4. **Conjecture. Watanabe-Yoshida, 2001.** Let $p > 2$ be and $d \geq 1$. Let $R_{p,d} = \overline{\mathbb{F}_p}[[x_0, x_1, \ldots, x_d]]/(x_0^2 + x_1^2 + \cdots + x_d^2)$. Let $(R, \mathfrak{m}, k)$ be an unmixed local ring of dimension d with $k = \overline{\mathbb{F}_p}$.
   (1) Let a_d be the coefficient of x^d in the Taylor series expansion of $\sec x + \tan x$. If R is not regular then $e_{HK}(R) \geq e_{HK}(R_{p,d}) \geq 1 + a_d$.
   (2) If the first equality holds then $\widehat{R} \simeq R_{p,d}$.
5. **V. Trivedi, 2022.** The Watanabe-Yoshida conjecture (1) is true if $p > d - 2$.
6. **Goto-Nakamura, 2001.** Let R be unmixed and a homomorphic image of a CM local ring. Let I be a parameter ideal. Then $e_{HK}(I) \geq \ell(R/I^*)$.
   The equality holds iff R is F-rational. In particular R is CM and normal.

The HK multiplicity of powers of an ideal

- **Theorem.** Let R be a Cohen-Macaulay local ring and I be generated by a system of parameters. Then $I^{[q]}$ is also generated by an sop. Hence for all $n \geq 1$,
$$\ell\left(\frac{R}{(I^{[q]})^n}\right) = \ell\left(\frac{R}{I^{[q]}}\right)\binom{n+d-1}{d} = e(I)\, q^d \binom{n+d-1}{d}.$$
- Therefore, $e_i(I^{[q]}) = 0$ for all $i \geq 1$. Moreover, $e_{HK}(I^n) = e(I)\binom{n+d-1}{d}$.
- **Definition.** An ideal I in a Noetherian local ring is called a **stable ideal** if for any minimal reduction J of I, $JI = I^2$.
- **Example.** Recall that a CM local ring R has minimal multiplicity if $\operatorname{emb}(R) - \dim R + 1 = e(R)$, where $\operatorname{emb}(R) = \mu(\mathfrak{m})$.
- **Theorem.** R has minimal multiplicity if and only if $\mathfrak{m}$ is a stable ideal.
- **Theorem. (K.-i. Watanabe-K.-i. Yoshida, 2001)** Let R be a CM local ring of dimension $d \geq 2$ and I be a stable ideal. Then for all $n \geq 1$,
$$e_{HK}(I^n) = e(I)\binom{n+d-1}{d} - (e(I) - e_{HK}(I))\binom{n+d-2}{d-1}.$$

Ilya Smirnov's questions about $e_{HK}(I^n)$

- **Theorem.** **(V. Trivedi, 2017)** Let I be a an ideal generated by elements of same degree in a standard graded ring over a perfect field of characteristic p. Then

$$\lim_{k\to\infty} \frac{e_{HK}(I^k) - e(I^k)/d!}{k^{d-1}} = \frac{e(I)}{2(d-2)!} - \lim_{q\to\infty} \frac{e_1(I^{[q]})}{(d-1)!q^d}.$$

- **Theorem.** **(I. Smirnov, 2019)** If $(R, \mathfrak{m})$ is a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal, then

$$e_{HK}(I^n) = e(I)\binom{n+d-1}{d} - \lim_{q\to\infty} \frac{e_1(I^{[q]})}{q^d}\binom{n+d-2}{d-1} + O(n^{d-2}).$$

- **Question 1.** Does the limit $\lim_{q\to\infty} e_i(I^{[q]})/q^d$ exist for all i?
- **Question 2.** Is it true that for all large integers n,

$$e_{HK}(I^n) = \sum_{i=0}^{d} (-1)^i \binom{n+d-1-i}{d-i} \lim_{q\to\infty} \frac{e_i(I^{[q]})}{q^d}$$

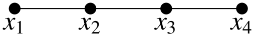
The HK multiplicity in face rings of simplicial complexes

- Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set $[r] = \{1, 2, \ldots, r\}$. Let $S = K[x_1, \ldots, x_r]$ be a polynomial ring over a field K. Let F be a facet of Δ and $\mathfrak{p}_F = (x_i \mid i \notin F)$.
- Let $F_1, F_2, \ldots, F_\alpha$ be the facets of Δ. Put $I_\Delta = \mathfrak{p}_{F_1} \cap \mathfrak{p}_{F_2} \cap \ldots \cap \mathfrak{p}_{F_\alpha}$.
- The face ring of Δ is $R = k[\Delta] = S/I_\Delta$ and $\dim k[\Delta] = d$.
- Let $\mathfrak{n} = \mathfrak{m}/I$ denote the maximal homogeneous ideal of R, where $\mathfrak{m}$ denotes the maximal homogeneous ideal of S.
- **(Banerjee-Goel-Verma, 2022)** Let R be a d-dimensional face ring of a simplicial complex. Let $J = (x_1^{v_1}, \ldots, x_r^{v_r})R$, where $v_i > 0$ for all i.
- Then the Hilbert-Kunz function $\ell(R/(J^{[q]})^k)$ is a polynomial for all $q, k \in \mathbb{N}$ and $\lim_{q\to\infty} e_i(J^{[q]})/q^d = 0$ for all $i = 1, \ldots, d$ and $k \geq 1$.
- The Hilbert-Kunz multiplicity is given by

$$e_{HK}(J^k) = e(J)\binom{k+d-1}{d}.$$

Example

- Let Δ be the 1-dimensional simplicial complex on 4 vertices.



- Set $\mathfrak{p}_1 = (x_3, x_4)$, $\mathfrak{p}_2 = (x_1, x_4)$ and $\mathfrak{p}_3 = (x_1, x_2)$. Let $I = \cap_{i=1}^{3} \mathfrak{p}_i$.
- Then $R = K[\Delta] = K[x_1, \ldots, x_4]/I = S/I$ is a 2-dim. CM graded ring.
- Let $\mathfrak{n}$ be the maximal homogeneous ideal of R. Then for all $q, k \geq 1$,

$$\ell\left(\frac{R}{\mathfrak{n}^{[q]k}}\right) = \sum_{i=1}^{3} \ell\left(\frac{S}{\mathfrak{p}_i + \mathfrak{m}^{[q]k}}\right) - \sum_{1 \leq i < j \leq 3} \ell\left(\frac{S}{\mathfrak{p}_i + \mathfrak{p}_j + \mathfrak{m}^{[q]k}}\right)$$

$$+ \ell\left(\frac{S}{\sum_{i=1}^{3} \mathfrak{p}_i + \mathfrak{m}^{[q]k}}\right)$$

$$= 3\binom{k+1}{2}q^2 - (2kq + 1) + 1 = 3\binom{k+1}{2}q^2 - 2kq.$$

$$\implies e(\mathfrak{n}^{[q]}) = 3q^2,\ e_1(\mathfrak{n}^{[q]}) = 2q,\ e_2(\mathfrak{n}^{[q]}) = 0,\ e_{HK}(\mathfrak{n}^k) = 3\binom{k+1}{2}.$$

An answer to Smirnov's question 2

- **Definition.** The **reduction number** $r(I)$ of an ideal I is the smallest integer n so that $JI^n = I^{n+1}$ where J varies of all minimal reductions.
- The **associated graded ring of *I*** is defined to be

$$G(I) = \bigoplus_{n=0}^{\infty} \frac{I^n}{I^{n+1}}.$$

- **Theorem. (Banerjee-Goel-Verma, 2022)** Let $(R, \mathfrak{m})$ be a CM local ring of dimension $d \geq 1$ and prime characteristic p. Let I be an $\mathfrak{m}$-primary ideal. If $\operatorname{depth} G(I^{[q]}) \geq d-1$ for all large q, then for all $i \geq 1$,

$$L_i(I) = \lim_{q \to \infty} \frac{e_i(I^{[q]})}{q^d} = \sum_{n=i}^{r(I)} \binom{n-1}{i-1} \left[e(I) - \sum_{j=0}^{d} (-1)^j \binom{d}{j} e_{HK}(I^{n-j}) \right].$$

- For all $n \geq r(I) - d + 1$, $e_{HK}(I^n) = \sum_{i=0}^{d} (-1)^i L_i(I) \binom{n+d-1-i}{d-i}$.

Example

- Let $R = \mathbb{F}_3[[x, y, z]]/(x^3 - y^3 z)$ and $\mathfrak{m} = (x, y, z)$.
- Then $I = (y, z)$ is a minimal reduction of $\mathfrak{m}$ and $r(\mathfrak{m}) = 2$.
- $G(\mathfrak{m}^{[q]})$ is Cohen-Macaulay for all large q.
- For all $k \geq 1$,

$$\begin{aligned} e_{HK}(\mathfrak{m}^k) &= e(\mathfrak{m}) \left[\binom{k+1}{2} - 2k + 1 \right] \\ &- (k-2)\, e_{HK}(\mathfrak{m}) + (k-1)\, e_{HK}(\mathfrak{m}^2). \end{aligned}$$

- Note that $e(\mathfrak{m}) = 3$ and $e_{HK}(\mathfrak{m}) = 3$ (A. Conca, 1996).
- In order to compute $e_{HK}(\mathfrak{m}^2)$, we compute

$$\ell\left(\frac{\mathbb{F}_3[[x, y, z]]}{(x^3 - y^3 z) + \mathfrak{m}^{2[q]}}\right) = \ell\left(\frac{\mathbb{F}_3[x, y, z]}{(x^3, y^{2q}, y^q z^q, z^{2q})}\right) = 9q^2.$$

Therefore, $e_{HK}(\mathfrak{m}^2) = 9$.

HK multiplicity of powers using depth $G(I^{[q]})$

1. **Theorem.** **(Huckaba-Marley, 1997)** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let I be an $\mathfrak{m}$-primary ideal with a minimal reduction $J = (a_1, a_2, \ldots, a_d)$. Let $r = r(I)$ and depth $G(I) \geq d - 1$. Then for all $n \in \mathbb{Z}$,
   1. $\Delta^d[P_I(n) - H_I(n)] = \ell(I^n/JI^{n-1})$.
   2. $\Delta^d P_I(n) = e_0(I)$ and $\Delta^d(H_I(n)) = \sum_{j=0}^{d}(-1)^j\binom{d}{j}\ell(R/I^{n-j})$.
   3. $e_i(I) = \sum_{n=i}^{r}\binom{n-1}{i-1}\ell(I^n/JI^{n-1})$ for $i = 1, 2, \ldots, d$.
2. **Theorem.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ and prime characteristic p. Let I be an $\mathfrak{m}$-primary ideal with a minimal reduction $J = (a_1, a_2, \ldots, a_d)$. Let $r = r(I)$ and depth $G(I^{[q]}) \geq d - 1$ for all large q. Then
   1. For all $i = 1, 2, \ldots, d$ the limit $L_i(I) := \lim_{q\to\infty} e_i(I^{[q]})/q^d$ exists and
   $$L_i(I) = \lim_{q\to\infty}\frac{e_i(I^{[q]})}{q^d} = \sum_{n=i}^{r}\binom{n-1}{i-1}\left[e_0(I) - \sum_{j=0}^{d}(-1)^j\binom{d}{j}e_{HK}(I^{n-j})\right].$$
   2. For all $n \geq r - d + 1$, $\ell\left(\frac{R}{(I^{[q]})^n}\right) = \sum_{i=0}^{d}(-1)^i e_i(I^{[q]})\binom{n+d-1-i}{d-i}$.
   3. Hence for all $n \geq r - d + 1$, $e_{HK}(I^n) = \sum_{i=0}^{d}(-1)^i L_i(I)\binom{n+d-1-i}{d-i}$.

1. **Proof.** Since $JI^r = I^{r+1}$, $J^{[q]}(I^{[q]})^r = (I^{[q]})^{r+1}$. Hence $r(I^{[q]}) \leq r$ for all q.
2. Since depth $G(I^{[q]}) \geq d-1$ for all large q, $P_{I^{[q]}}(n) = H_{I^{[q]}}(n)$ for all $n \geq r-d+1$. For all $i = 1, 2, \ldots, d$,

$$e_i(I^{[q]}) = \sum_{n=i}^{r} \binom{n-1}{i-1} \ell\left(\frac{(I^{[q]})^n}{J^{[q]}(I^{[q]})^{n-1}}\right).$$

3. Now find $\beta_I(n) := \ell(I^n/JI^{n-1})$. for all n $\Delta^d(P_I(n) - H_I(n)) = \ell(I^n/JI^{n-1})$.

$$\beta_I(n) = e_0(I) - \Delta^d(H_I(n)) = e_0(I) - \sum_{j=0}^{d}(-1)^j \binom{d}{j} \ell(R/I^{n-j}).$$

4. Since depth $G(I^{[q]}) \geq d-1$ for large q, we have

$$\beta_{I^{[q]}}(n) = q^d e_0(I) - \Delta^d(H_{I^{[q]}}(n)) = q^d e_0(I) - \sum_{j=0}^{d}(-1)^j \binom{d}{j} \ell(R/(I^{[q]})^{n-j}).$$

5. Now find the limit: $\lim_{q\to\infty} \frac{\beta_{I^{[q]}}(n)}{q^d} = e_0(I) - \sum_{j=0}^{d}(-1)^j \binom{d}{j} e_{HK}(I^{n-j})$.

1. Therefore we have

$$L_i(I) = \lim_{q\to\infty} \frac{e_i(I^{[q]})}{q^d} = \sum_{n=i}^{r} \binom{n-1}{i-1} \left[e_0(I) - \sum_{j=0}^{d} (-1)^j \binom{d}{j} e_{HK}(I^{n-j}) \right].$$

2. Since $r(I^{[q]}) \leq r$, for all $n \geq r-d+1$, $H_{I^{[q]}}(n) = P_{I^{[q]}}(n)$. Hence for all $n \geq r-d+1$,

$$\ell(R/(I^{[q]})^n) = \sum_{i=0}^{d} (-1)^i e_i(I^{[q]}) \binom{n+d-1-i}{d-i}.$$

3. Divide by q^d and take the limit to see that for all $n \geq r-d+1$,

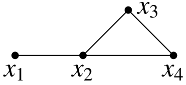
$$e_{HK}(I^n) = \sum_{i=0}^{d} (-1)^i L_i(I) \binom{n+d-1-i}{d-i}.$$

Smirnov's Conjecture about stable ideals

- **(D. G. Northcott, 1960)** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring local and I be an $\mathfrak{m}$-primary ideal. Then
$$e_0(I) - e_1(I) \leq \ell(R/I).$$
- **(C. Huneke, A. Ooishi, 1987)** Equality holds above $\iff$ I is stable.
- Stable ideals CM local rings have many interesting properties.
- We know their Hilbert polynomial. Their Rees algebra, associated graded ring and the fiber cone are CM.
- **Theorem. (Lipman-Teissier, 1979)** All complete ideals in regular local rings of dimension two are stable.
- **Conjecture. (I. Smirnov, 2019)** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal. Then
$$I \text{ is stable} \iff \lim_{q\to\infty} \frac{e_1(I^{[q]})}{q^d} = e(I) - e_{HK}(I).$$

A counter-example to Smirnov's conjecture

- Let Δ be the simplicial complex



Then $R = K[x_1, x_2, x_3, x_4]/((x_3, x_4) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_1, x_2))$ is the face ring of Δ and is a 2-dimensional Cohen-Macaulay ring.

- Let $\mathfrak{n}$ denote the maximal homogeneous ideal of R. Then

$$4 = e(\mathfrak{n}) = e_{HK}(\mathfrak{n}) \text{ and } \lim_{q \to \infty} e_1(\mathfrak{n}^{[q]})/q^2 = 0.$$

- Claim: $\mathfrak{n}$ is not stable. Using Stanley's formula, The Hilbert series of R

$$H(R, z) = \frac{(1 + 2z + z^2)}{(1 - z)^2}.$$

- As R is Cohen-Macaulay and degree of the numerator is $r(\mathfrak{n}) = 2$.

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