Projective Closure of Numerical Semigroups and Affine Monomial Curves

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Abstract

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A Numerical Semigroup Ring is the coordinate ring of an affine monomial curve, which is always Cohen-Macaulay. However, under projective closure, most of the good properties are not preserved any more. In this talk, we will discuss some such questions and present some results answering such questions. This talk is based on work with Joydip Saha, Ranjana Mehta, Pranjal Srivastava, Om Prakash Bhradwaj and Kriti Goel.

Preliminaries and some Questions

Preliminaries and some Questions

Let us assume that k is a field of characteristics 0.

Let $R = k[x_1, x_2, \ldots, x_n]$ denote the polynomial ring over the field k.

The Main Question (Kronecker?) By the Hilbert basis theorem, every $\mathfrak{p}$ is always finitely generated. By the Krull's theorem, each $\mathfrak{p}$ of height $n-1$ requires at least $n-1$ generators. A natural question is, for a fixed n, whether there is an upper bound on the number of elements in a minimal generating set of $\mathfrak{p}$.

The answer is **NO**, in general.

Preliminaries and some Questions

- **(Macaulay; 1916).** A family of prime ideals $\{P_r\}$ in the polynomial ring $k[x, y, z]$ which need at least r generators. Almost all the ideals define irreducible curves in $\mathbb{A}_k^3$, with singularities at the origin.
- **(Abhyankar; 1972).** On Macaulay's Example: Assuming that k is algebraically closed, this work gives a detailed proof of Macaulay's claims.
- **(Moh 1974; 1979).** Macaulay's example can not be adopted directly in the power series ring $k[[x, y, z]]$. Moh has given a family of prime ideals $\{P_r\}$ in $k[[x, y, z]]$, which need at least r generators, in fact exactly $r + 1$ generators. These ideals define rational irreducible curves in $\mathbb{A}_k^3$, which are analytically irreducible at the origin. Moreover, each $\{P_r\}$ is determinantal.
- **(Bresinsky; 1975).** Let $a \geq 4$ be even; set $m_1 = a(a+1)$, $m_2 = (a+1)(a-1)$, $m_3 = a(a+1) + (a-1)$, $m_4 = a(a-1)$ and $\gcd(m_1, m_2, m_3, m_4) = 1$. Take the parametrization $x_i = t^{m_i}$. In this case the set $\{\mu(\mathfrak{p}_a) \mid a \geq 4 \quad \text{even}\}$ is unbounded.

Preliminaries and some Questions

The Value Semigroup.

"In a conversation about [4], O. Zariski indicated to the author that there should be a relation between Gorenstein rings and symmetric value-semigroups, possibly allowing a new proof for a result of Herzog on complete intersections. In the following note it is shown that this is the case." - *The Value-Semigroup of a One-Dimensional Gorenstein Ring*, E. Kunz, Proceedings of the American Mathematical Society, Vol. 25, No. 4 (Aug., 1970), pp. 748–751.

Theorem (Kunz; 1970)
The ring R is Gorenstein if and only if its value semigroup $v(R)$ is symmetric.

Numerical Semigroups and Affine Monomial Curves

Numerical Semigroups and Affine Monomial Curves

Let $\mathbb{N} := \{0, 1, 2, \rightarrow\}$.

Definition
A subset S of $\mathbb{N}$ is a *submonoid of* $\mathbb{N}$ if it satisfies the following properties:

- If $0 \in S$;
- if $a, b \in S$ then $a + b \in S$.

Definition
A submonoid S of $\mathbb{N}$ is called a *numerical semigroup* if $\mathbb{N} \setminus S$ is a finite set.

Example

1. The set $\{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ is a submonoid of $\mathbb{N}_0$.

Numerical Semigroups and Affine Monomial Curves

One can prove some properties with some amount of effort:

Theorem

1. *Every numerical semigroup has a unique minimal generating set.*
2. *A submonoid S of $\mathbb{N}$ is a numerical semigroup if and only if* $\gcd(S) = 1$.
3. *Every nontrivial submonoid of $\mathbb{N}$ is isomorphic to a unique numerical semigroup.*

Numerical Semigroups and Affine Monomial Curves

If $m_0, \ldots, m_{e-1}$ are the denominations (positive integers) given, then, any amount M is realizable through these denominations if M can be written as $M = a_0 m_0 + \cdots + a_{e-1} m_{e-1}$, such that $a_0, a_1, \ldots, a_{e-1}$ are nonnegative integers.

Let

$$S = \{a_0 m_0 + \cdots + a_{e-1} m_{e-1} \mid a_0, \ldots, a_{e-1} \text{ nonnegative integers}\}.$$

The Coin Exchange Problem. Does there exist an integer $M \in \mathbb{N}_0 \setminus S$, such that every integer from $M + 1$ and higher belong to S?

Numerical Semigroups and Affine Monomial Curves

1. You can easily verify that S is a submonoid of $\mathbb{N}$.
2. With $\gcd(m_0, \ldots, m_{e-1}) = 1$ condition, this is also a numerical semigroup of $\mathbb{N}$.
3. Therefore, $\mathbb{N} \setminus S$ is finite and a maximum value (integer) exits which can not be realized through the denominations $m_0, \ldots, m_{e-1}$.

Therefore, the coin exchange problem has an affirmative answer. However, the hard part is to calculate the integer M in terms of $m_0, \ldots, m_{e-1}$.

Numerical Semigroups and Affine Monomial Curves

There exists an integer $c \in S$, such that, $c + i \in S$ for all $i \in \mathbb{N}$, but $c - 1 \notin S$. The integer $c - 1$ is called the *Frobenius number* and c is called the *conductor* of the semigroup S. We denote the Frobenius number by $F(S)$ and the conductor by $c(S)$.

The semigroup S is said to be *symmetric* if $[0, c-1] \cap S$ and $[0, c-1] \cap (\mathbb{N} \setminus S)$ have the same number of elements.

The number of elements in the finite set $\mathbb{N} \setminus S$ is called the *genus* of S, denoted by $g(S)$.

The number $n(S)$ is the number of elements $s \in S$ such that $s < F(S)$.

Numerical Semigroups and Affine Monomial Curves

Linear Diophantine Problem of Frobenius. Compute $F(S)$ and $g(S)$ in terms of $m_0, \ldots, m_{e-1}$.

Theorem (Sylvester; 1884)
Let $a < b$ be relatively prime positive integers and S the numerical semigroup generated by a, b. Then

1. $F(S) = ab - a - b$,
2. $g(S) = \frac{(a-1)(b-1)}{2}$.

Numerical Semigroups and Affine Monomial Curves

Theorem (Selmer; 1977)
Let S be a numerical semigroup and let $0 \neq n \in S$. Let $\mathcal{B} := \mathcal{A}p(S,n) := \{0 = w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element of S congruent to i modulo n. The set $\mathcal{A}p(S,n)$ is called the Apéry set of the numerical semigroup S with respect to n.Then,

1. $F(S) = \max(\mathcal{A}p(S,n)) - n$,
2. $g(S) = \frac{1}{n}\left(\sum_{w\in \mathcal{A}p(S,n)} w\right) - \frac{n-1}{2}$.

Numerical Semigroups and Affine Monomial Curves

Example

1. The set $S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ is a numerical semigroup with the unique minimal generating set $\{5, 7, 9\}$. The largest number which does not belong to S is 13.
2. The set $S = \{0, 4, 7, 8, 10, 11, 12, 14, 16, \rightarrow\}$ is a numerical semigroup with the minimal generating set $\{4, 7, 10\}$. The largest number which does not belong to S is 15.

Can you spot a distinguishing difference between these two examples?

Conjecture [Wilf's conjecture] $F(S) + 1 \leq e \cdot n(S)$.

Suppose that S is a numerical semigroup generated by the sequence $\mathbf{m} = (m_1, \ldots, m_e)$.

Let $\mathcal{C}(\mathbf{m})$ be the affine curve defined by the monomial parametrization $x_i = t^{m_i}$, $1 \leq i \leq e$.

Let $\mathfrak{p}(\mathbf{m})$ denote the defining ideal of the curve, which is the kernel of the affine algebra map from $k[x_1, \ldots, x_e]$ to $k[t]$ defined by $x_i \mapsto t^{m_i}$, $1 \leq i \leq e$.

Question. Given e, if S is symmetric of embedding dimension e, then is $\mu(\mathfrak{p}(\mathbf{m}))$ a bounded function of e? What about the higher Betti numbers?

The answer is yes if $e \leq 4$. The question is open is general for $e \geq 5$.

Numerical Semigroups and Affine Monomial Curves

Some important examples for our study.

Bresinsky's example. Let $h \geq 2$. Bresinsky defined the following numerical semigroup $\Gamma((2h-1)2h, (2h-1)(2h+1), 2h(2h+1), 2h(2h+1)+2h-1)$.

Backelin's example. Backelin defined the following numerical semigroups $\Gamma(s, s+3, s+3n+1, s+3n+2)$, for $n \geq 2$, $r \geq 3n+2$ and $s = r(3n+2)+3$.

Arslan's example. Arslan defined the following numerical semigroups $\Gamma(h(h+1), h(h+1)+1, (h+1)^2, (h+1)^2+1)$.

One common thing about all three examples is that all have unboundedness of minimal number of generators of the defining ideal (the first Betti number).

Numerical Semigroups and Affine Monomial Curves

Numerical Semigroups defined by Concatenation of Arithmetic Sequences.

Let $e \geq 4$. Consider the sequence of positive integers $a < a+d < a+2d < \ldots < a+(r-1)d < b < b+d < \ldots < b+(s-1)d$, such that,

- integers $r, s \geq 2$,
- $r + s = e$,
- $\gcd(a, d) = 1$ and d does not divide $b - a$,
- the sequence is minimal.

Γ is called the *numerical semigroup generated by concatenation of two arithmetic sequences with the same common difference d.*

The minimality assumption can not be dropped. For example, $e = d = 4$, $a = 5$ and $b = 10$ gives us $5, 9, 10, 14$, which is not minimal.

Numerical Semigroups and Affine Monomial Curves

Bresinsky's examples fall into the category of numerical semigroups defined by concatenation. We have studied another family, defined by concatenation, in arbitrary embedding dimension e.

Example. Let $e \geq 4$, $n \geq 5$ and $q \geq 0$. Let us define $m_i := n^2 + (e-2)n + q + i$, for $0 \leq i \leq e-3$ and $m_{e-2} := n^2 + (e-1)n + q + (e-3)$, $m_{e-1} := n^2 + (e-1)n + q + (e-2)$. Let $\mathfrak{S}_{(n,e,q)} = \langle m_0, \ldots, m_{e-1} \rangle$, then $\{m_0, \ldots, m_{e-1}\}$ is a minimal generating set for the semigroup $\mathfrak{S}_{(n,e,q)}$. Let $\mathcal{Q}_{(n,e,q)}$ denote the defining prime ideal for the monomial curve defined the parametrization $x_i = t^{m_i}$.

Theorem (Mehta-Saha-S). $\mu(\mathcal{Q}_{(n,4,0)}) = 2(n+1)$.

Conjecture (Mehta-Saha-S). The set $\{\mu(\mathcal{Q}_{(n,e,q)}) \mid n \geq 5, e \geq 4, q \geq 0\}$ is unbounded above.

Projective Closure of Affine Monomial Curves

Projective Closure of Affine Monomial Curves

Let $m_1, \ldots, m_e$ be a sequence of distinct positive integers, with $m_e > m_i$ for all $i < e$. We fix $m_0 = 0$.

We define the semigroup $\overline{\Gamma(m_1, \ldots m_e)} \subset \mathbb{N}^2$, generated by $\{(m_i, m_e - m_i) \mid 0 \leq i \leq e\}$.

Let $\overline{\mathfrak{p}(m_1, \ldots m_e)}$ be the kernel of k-algebra map $\eta^H : k[x_0, \ldots, x_e] \longrightarrow k[a, b]$, $\eta^H(x_i) = a^{m_e - m_i} b^{m_i}$, $i \geq 0$.

The homogenization of the ideal $\mathfrak{p}(m_1, \ldots, m_e)$, w.r.t. the variable x_0 is $k[x_0, \ldots, x_e]$. Thus the projective curve $\{[(a^{m_e} : a^{m_e - m_1} b^{m_1} : \cdots : b^{m_e})] \in \mathbb{P}^e_k \mid a, b \in k\}$ is the projective closure of the affine curve $C(m_1, \ldots, m_e) := \{(b^{m_1}, \ldots b^{m_e}) \in \mathbb{A}^e_k \mid b \in k\}$. We denote this by $\overline{C(m_1, \ldots, m_e)}$.

The k-algebra $k[x_0, \ldots, x_e]/\overline{\mathfrak{p}(m_1, \ldots m_e)}$ is called the projective numerical semigroup ring.

Cohen-Macaulayness and Syzygies of the three Examples

Cohen-Macaulayness and Syzygies of the three Examples

Bresinsky curves. Herzog and Stamate have proved that the projecive closures of the Bresinsky curves are not arithmetically Cohen-Macaulay, using a Gröbner basis criterion. Stamate has proved that the Betti sequence of the affine Bresinsky curves is given by $(1, 4h, 8h-4, 4h-3)$.

Theorem (Saha-S-Srivastava). For $h \geq 2$, a minimal graded free resolution of the homogenized ideal $\overline{\mathfrak{Q}_h}$, which is the defining ideal of the projective closure of the Bresinsky curves, over the polynomial ring $k[x_0, x_1, x_2, x_3, x_4]$, is given by

$$\mathfrak{B}_{\mathbf{h}} : 0 \longrightarrow R \xrightarrow{\mathfrak{B}_h^4} R^{4h+3} \xrightarrow{\mathfrak{B}_h^3} R^{8h+4} \xrightarrow{\mathfrak{B}_h^2} R^{4h+3} \xrightarrow{\mathfrak{B}_h^1} R \longrightarrow R/\overline{\mathfrak{Q}_h} \longrightarrow 0,$$

where the matrices $\mathfrak{B}^i, 1 \leq i \leq 4$ have been defined above.

Proof. A Gröbner basis computation along with the Buchsbaum-Eisenbud acyclicity criterion. $\square$

Cohen-Macaulayness and Syzygies of the three Examples

Backelin curves.

Theorem (Saha-S-Srivastava). The projective closure $\overline{B}_{nr}$ of the Backelin curves is arithmetically Cohen-Macaulay.

Theorem (Saha-S-Srivastava). For $n \geq 4$, a minimal graded free resolution of the defining ideal of the projective closure of the Backelin curves $\overline{\Gamma}_{nr}$ is given by

$$\overline{\mathfrak{R}_h} : 0 \to R^{3n+2} \xrightarrow{\mathfrak{C}_{\mathfrak{h}}^{3}} R^{6n+5} \xrightarrow{\mathfrak{C}_{\mathfrak{h}}^{2}} R^{3n+4} \xrightarrow{\mathfrak{C}_{\mathfrak{h}}^{1}} R \to R/\overline{\mathfrak{P}_{\mathfrak{nr}}} \to 0,$$

where the matrices $\mathfrak{C}_{\mathfrak{h}}^{i}, 1 \leq i \leq 3$ are defined above. Moreover, for $n \geq 4$, a minimal graded free resolution of the defining ideal of the Backelin curves Γ_{nr} is given by,

$$\mathfrak{R}_h : 0 \to R^{3n+2} \xrightarrow{\tilde{\mathfrak{C}}_{\mathfrak{h}}^{3}} R^{6n+5} \xrightarrow{\tilde{\mathfrak{C}}_{\mathfrak{h}}^{2}} R^{3n+4} \xrightarrow{\tilde{\mathfrak{C}}_{\mathfrak{h}}^{1}} R \to R/\mathfrak{P}_{\mathfrak{nr}} \to 0,$$

where the matrices $\tilde{\mathfrak{C}}_{\mathfrak{h}}^{i}, 1 \leq i \leq 3$, are obtained by evaluating $x_0 = 1$ in $\mathfrak{C}_{\mathfrak{h}}^{i}, 1 \leq i \leq 3$.

We have also computed the Hilbert series.

Cohen-Macaulayness and Syzygies of the three Examples

Arslan curves. It was proved that the Arslan curves are arithmetically Cohen-Macaulay, using Gröbner basis. Stamate has proved that the Betti sequence of the corresponding affine monomial curve is $(1, 2h+2, 4h, 2h-1)$.

Theorem (Saha-S-Srivastava). For $h \geq 2$, a graded free minimal resolution of $\overline{\mathfrak{I}_h}$, the defining ideal of the projective closure of the Arslan curves $\mathfrak{A}_h$, is

$$\mathfrak{M}_{\mathbf{h}} : \ 0 \longrightarrow R^{2h-1} \xrightarrow{\mathfrak{A}_h^3} R^{4h+1} \xrightarrow{\mathfrak{A}_h^2} R^{2h+3} \xrightarrow{\mathfrak{A}_h^1} R \longrightarrow R/\overline{\mathfrak{I}_h} \longrightarrow 0,$$

where the matrices $\mathfrak{A}_h^i$, $1 \leq i \leq 3$, are defined above.

Gluing, the Gorenstein property and the Betti sequence

Gluing, the Gorenstein property and the Betti sequence

Question 1. Suppose the projective closures of two affine monomial curves are arithmetically Cohen-Macaulay (respectively Gorenstein). Which condition(s) on gluing does preserve the arithmetically Cohen-Macaulay (respectively Gorenstein) property of the projective closure of the monomial curve obtained by gluing of these two monomial curves?

Question 2. When can we say that the Betti numbers of the affine monomial curve and its projective closure are the same?

Question 3. What is the connection between the Cohen-Macaulayness of the tangent cone at the origin and the arithmetically Cohen-Macaulayness of the projective closure?

Gluing, the Gorenstein property and the Betti sequence

Gluing. Let $\Gamma_1 = \Gamma(m_1, \ldots, m_l)$ and $\Gamma_2 = \Gamma(n_1, \ldots, n_k)$ be two numerical semigroups, with $m_1 < \cdots < m_l$ and $n_1 < \cdots < n_k$. Let $p = b_1 m_1 + \cdots + b_l m_l \in \Gamma_1$ and $q = a_1 n_1 + \cdots + a_k n_k \in \Gamma_2$ be two positive integers satisfying $\gcd(p, q) = 1$, with $p \notin \{m_1, \ldots, m_l\}$, $q \notin \{n_1, \ldots, n_l\}$ and $\{qm_1, \ldots, qm_l\} \cap \{pn_1, \ldots, pn_k\} = \phi$. The numerical semigroup $\Gamma_1 \#_{p,q} \Gamma_2 = \langle qm_1, \ldots, qm_l, pn_1, \ldots, pn_k \rangle$ is called a *gluing* of the semigroups Γ_1 and Γ_2.

Star Gluing. The numerical semigroup $\Gamma_1 \#_{p,q} \Gamma_2$, obtained by gluing of $\Gamma_1 = \Gamma(m_1, \ldots, m_l)$ and $\Gamma_2 = \Gamma(n_1, \ldots, n_k)$, with respect to the positive integers p and q, is called a *star gluing* if $p = b_l m_l \in \Gamma_1$ and $q = a_1 n_1 + a_2 n_2 + \cdots + a_k n_k \in \Gamma_2$, with $a_1 + a_2 + \cdots + a_k \leq b_l$.

It is important to find the largest integer from the generator of the numerical semigroup $\Gamma_1 \star_{p,q} \Gamma_2$, obtained by star gluing.

Gluing, the Gorenstein property and the Betti sequence

Theorem (Saha-S-Srivastava). Let Γ_1 and Γ_2 be numerical semigroups such that the associated projective closures $\overline{\mathcal{C}(\Gamma_1)}$ and $\overline{\mathcal{C}(\Gamma_2)}$ are arithmetically Cohen-Macaulay. Let $\Gamma = \Gamma_1 \star_{p,q} \Gamma_2$ be a star gluing of Γ_1 and Γ_2. Then the projective closure $\overline{\mathcal{C}(\Gamma)}$ associated to Γ is arithmetically Cohen-Macaulay.

Theorem (Saha-S-Srivastava). Let Γ_1 and Γ_2 be numerical semigroups such that the associated projective closures $\overline{\mathcal{C}(\Gamma_1)}$ and $\overline{\mathcal{C}(\Gamma_2)}$ are Gorenstein. Let $\Gamma = \Gamma_1 \star_{p,q} \Gamma_2$ be a star gluing of Γ_1 and Γ_2. Then the projective closure $\overline{\mathcal{C}(\Gamma)}$ associated to Γ is Gorenstein.

Proof uses the notion of a *good* Apéry set due to Cavaliere and Niesi.

Gluing, the Gorenstein property and the Betti sequence

Theorem (Saha-S-Srivastava). Let Γ be a numerical semigroup, such that $\overline{\mathcal{C}(\Gamma)}$ is arithmetically Cohen-Macaulay. If there exists a Gröbner basis G of the defining ideal $\mathfrak{p}(\Gamma)$ of $\mathcal{C}(\Gamma)$, with respect to the degree reverse lexicographic ordering $x_i > x_e > x_0$, for $i = 1, \ldots, e-1$, such that x_e belongs to the support of all non-homogeneous elements of G, then $\beta_i^R(C(\Gamma)) = \beta_i^S(\overline{C(\Gamma)})$, where $R = k[x_1, \ldots, x_e]$ and $S = k[x_0, x_1, \ldots, x_e]$.

Corollary. Let $\Gamma = \Gamma(m_1, \ldots, m_e)$ be a numerical semigroup, minimally generated by an arithmetic sequence $m_1 < m_2 < \cdots < m_e$, such that $m_i = m_1 + (i-1)d, 1 \leq i \leq e$, and $m_1 = q(n-1) + r, r \in [1, e]$. Then $\beta_i^R(C(\Gamma)) = \beta_i^S(\overline{C(\Gamma)})$.

Gluing, the Gorenstein property and the Betti sequence

Theorem (Saha-S-Srivastava). Let Γ be a numerical semigroup minimality generated by $m_1 < \cdots < m_e$. Suppose $\Gamma_1 = \langle c\Gamma, d\rangle$ is a simple gluing (Γ glued with $\mathbb{N}$), where $c > 1$ and d are co-prime integers with $d \in \Gamma \setminus \{m_1, \ldots, m_e\}$. If $\overline{\mathcal{C}(\Gamma)}$ is arithmetically Cohen-Macaulay, then, $\overline{\mathcal{C}(\Gamma_1)}$ is arithmetically Cohen-Macaulay.

Corollary. Let Γ and Γ_1 be as above. Then $\overline{\mathcal{C}(\Gamma)}$ and $\overline{\mathcal{C}(\Gamma_1)}$ have the same type. In particular, if $\overline{\mathcal{C}(\Gamma)}$ is Gorenstein then $\overline{\mathcal{C}(\Gamma_1)}$ is Gorenstein.

It can be proved that $\beta_i^S(\overline{C(\Gamma_1)}) = \beta_i^S(\overline{C(\Gamma)}) + \beta_{i-1}^S(\overline{C(\Gamma)})$ using a mapping cone technique. This is the main observation for proving the above results and it is indeed the projective version of the result obtained by Gimenez and Srinivasan for the affine case.

Wilf’s Conjecture

Wilf's Conjecture

Extension of Wilf's Conjecture. Let S be a $\mathcal{C}$-semigroup ($\mathcal{H}(S) = (\text{cone}(S) \setminus S) \cap \mathbb{N}^d$ is nonempty and finite). The extended Wilf's conjecture is

$$|\{g \in S \mid g < F(S)_<\}| \cdot e(S) \geq \mathcal{N}(F(S)_<) + 1,$$

where $<$ is a term order and $e(S)$ denotes the embedding dimension of S.

Theorem (Bharadwaj-Goel-S). Let S be a $\mathcal{C}$-semigroup such that $\text{cone}(S) \cap \mathbb{N}^d = \mathbb{N}^d$, $d > 1$. If S is a $<$-symmetric or a $<$-pseudo-symmetric semigroup, then the extended Wilf's conjecture holds.

Unfortunately, the semigroup in $\mathbb{N}^2$, defining the projective closure of a numerical semigroup γ, is not a $\mathcal{C}$-semigroup. One can ask a somewhat broad question:

Question. Let Γ be a numerical semigroup which satisfies the Wilf conjecture. What can one say about the semigroup in $\mathbb{N}^2$, defining the projective closure?

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Thank you!