A cohomological version of the singular Hodge conjecture

Inder Kaur

Goethe Universität, Frankfurt am Main, Germany

July 18, 2022

Notations

- Always over $\mathbb{C}$
- A *(pure) Hodge structure of weight* $r \in \mathbb{Z}$ denoted $(V_{\mathbb{Z}}, V^{p,q})$, consists of:
  - A finitely generated free abelian group (lattice) $V_{\mathbb{Z}}$,
  - a decomposition $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$, of the complexification $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, satisfying $V^{p,q} = \overline{V^{p,q}}$.
- Equivalently we can define a pure Hodge structure on $V_{\mathbb{C}}$ by giving a filtration

$$V_{\mathbb{C}} = F^0 \subset F^1 \subset \dots F^n \subset 0$$

such that $V_{\mathbb{C}} \simeq F^p \oplus \overline{F^{n-p+1}}$.

Notations

- Always over $\mathbb{C}$
- A *(pure) Hodge structure of weight $r \in \mathbb{Z}$* denoted $(V_{\mathbb{Z}}, V^{p,q})$, consists of:
  - A finitely generated free abelian group (lattice) $V_{\mathbb{Z}}$,
  - a decomposition $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$, of the complexification $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, satisfying $V^{p,q} = \overline{V^{p,q}}$.
- Equivalently we can define a pure Hodge structure on $V_{\mathbb{C}}$ by giving a filtration

$$V_{\mathbb{C}} = F^0 \subset F^1 \subset \dots F^n \subset 0$$

such that $V_{\mathbb{C}} \simeq F^p \oplus \overline{F^{n-p+1}}$.

Notations

- Always over $\mathbb{C}$
- A *(pure) Hodge structure of weight* $r \in \mathbb{Z}$ denoted $(V_{\mathbb{Z}}, V^{p,q})$, consists of:
  - A finitely generated free abelian group (lattice) $V_{\mathbb{Z}}$,
  - a decomposition $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$, of the complexification $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, satisfying $V^{p,q} = \overline{V^{p,q}}$.
- Equivalently we can define a pure Hodge structure on $V_{\mathbb{C}}$ by giving a filtration

$$V_{\mathbb{C}} = F^0 \subset F^1 \subset \dots F^n \subset 0$$

such that $V_{\mathbb{C}} \simeq F^p \oplus \overline{F^{n-p+1}}$.

Notations

- Always over $\mathbb{C}$
- A *(pure) Hodge structure of weight* $r \in \mathbb{Z}$ denoted $(V_{\mathbb{Z}}, V^{p,q})$, consists of:
  - A finitely generated free abelian group (lattice) $V_{\mathbb{Z}}$,
  - a decomposition $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$, of the complexification $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, satisfying $V^{p,q} = \overline{V^{p,q}}$.
- Equivalently we can define a pure Hodge structure on $V_{\mathbb{C}}$ by giving a filtration

$$V_{\mathbb{C}} = F^0 \subset F^1 \subset \dots F^n \subset 0$$

such that $V_{\mathbb{C}} \simeq F^p \oplus \overline{F^{n-p+1}}$.

Notations

- Always over $\mathbb{C}$
- A *(pure) Hodge structure of weight* $r \in \mathbb{Z}$ denoted $(V_{\mathbb{Z}}, V^{p,q})$, consists of:
  - A finitely generated free abelian group (lattice) $V_{\mathbb{Z}}$,
  - a decomposition $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$, of the complexification $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, satisfying $V^{p,q} = \overline{V^{p,q}}$.
- Equivalently we can define a pure Hodge structure on $V_{\mathbb{C}}$ by giving a filtration

$$V_{\mathbb{C}} = F^0 \subset F^1 \subset \dots F^n \subset 0$$

such that $V_{\mathbb{C}} \simeq F^p \oplus \overline{F^{n-p+1}}$.

Cycle class map

- X smooth, projective variety of dimension n. Then $H^{2p}(X, \mathbb{Q})$ admits the Hodge decomposition.
- For $r = 2p$, elements of $H^{p,p}(X, \mathbb{Z}) := H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X, \mathbb{C})$ are called *Hodge classes*.
- An *algebraic cycle* of codimension p is a formal linear combination of irreducible codimension p subvarieties of X.
- $Z^p(X) :=$ free abelian group of codimension p algebraic cycles of X.
- The *cycle class map*:

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, Q)$$

Cycle class map

- X smooth, projective variety of dimension n. Then $H^{2p}(X, \mathbb{Q})$ admits the Hodge decomposition.
- For $r = 2p$, elements of $H^{p,p}(X, \mathbb{Z}) := H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X, \mathbb{C})$ are called *Hodge classes*.
- An *algebraic cycle* of codimension p is a formal linear combination of irreducible codimension p subvarieties of X.
- $Z^p(X)$:= free abelian group of codimension p algebraic cycles of X.
- The *cycle class map*:

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, Q)$$

Cycle class map

- X smooth, projective variety of dimension n. Then $H^{2p}(X, \mathbb{Q})$ admits the Hodge decomposition.
- For $r = 2p$, elements of $H^{p,p}(X, \mathbb{Z}) := H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X, \mathbb{C})$ are called *Hodge classes*.
- An *algebraic cycle* of codimension p is a formal linear combination of irreducible codimension p subvarieties of X.
- $Z^p(X):=$ free abelian group of codimension p algebraic cycles of X.
- The *cycle class map*:

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, Q)$$

Cycle class map

- X smooth, projective variety of dimension n. Then $H^{2p}(X, \mathbb{Q})$ admits the Hodge decomposition.
- For $r = 2p$, elements of $H^{p,p}(X, \mathbb{Z}) := H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X, \mathbb{C})$ are called *Hodge classes*.
- An *algebraic cycle* of codimension p is a formal linear combination of irreducible codimension p subvarieties of X.
- $Z^p(X)$:= free abelian group of codimension p algebraic cycles of X.
- The *cycle class map*:

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, Q)$$

Cycle class map

- X smooth, projective variety of dimension n. Then $H^{2p}(X, \mathbb{Q})$ admits the Hodge decomposition.
- For $r = 2p$, elements of $H^{p,p}(X, \mathbb{Z}) := H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X, \mathbb{C})$ are called *Hodge classes*.
- An *algebraic cycle* of codimension p is a formal linear combination of irreducible codimension p subvarieties of X.
- $Z^p(X)$:= free abelian group of codimension p algebraic cycles of X.
- The *cycle class map*:

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, Q)$$

Classical Hodge conjecture

- One can show: $\mathrm{Im}(c) \subset H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$ i.e., the image of the cycle class map are Hodge classes.
- **Hodge conjecture**: Does every rational (p, p) Hodge class comes from an algebraic cycle?
- Precisely: Is the cycle class map

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$$

surjective?
- True for $p = 1$: Lefschetz $(1, 1)$!
- very general abelian varieties (Recall, an element is *general* if it lies in the complement of finitely many proper closed subsets; very general if in the complement of countably many proper closed subsets)

Classical Hodge conjecture

- One can show: $\mathrm{Im}(c) \subset H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$ i.e., the image of the cycle class map are Hodge classes.
- **Hodge conjecture**: Does every rational (p, p) Hodge class comes from an algebraic cycle?
- Precisely: Is the cycle class map

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$$

surjective?
- True for $p = 1$: Lefschetz $(1, 1)$!
- very general abelian varieties (Recall, an element is *general* if it lies in the complement of finitely many proper closed subsets; very general if in the complement of countably many proper closed subsets)

Classical Hodge conjecture

- One can show: $\mathrm{Im}(c) \subset H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$ i.e., the image of the cycle class map are Hodge classes.
- **Hodge conjecture**: Does every rational (p, p) Hodge class comes from an algebraic cycle?
- Precisely: Is the cycle class map

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$$

surjective?

- True for $p = 1$: Lefschetz $(1, 1)$!
- very general abelian varieties (Recall, an element is *general* if it lies in the complement of finitely many proper closed subsets; very general if in the complement of countably many proper closed subsets)

Classical Hodge conjecture

- One can show: $\mathrm{Im}(c) \subset H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$ i.e., the image of the cycle class map are Hodge classes.
- **Hodge conjecture**: Does every rational (p, p) Hodge class comes from an algebraic cycle?
- Precisely: Is the cycle class map

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$$

surjective?
- True for $p = 1$: Lefschetz $(1, 1)$!
- very general abelian varieties (Recall, an element is *general* if it lies in the complement of finitely many proper closed subsets; very general if in the complement of countably many proper closed subsets)

Classical Hodge conjecture

- One can show: $\mathrm{Im}(c) \subset H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$ i.e., the image of the cycle class map are Hodge classes.
- **Hodge conjecture**: Does every rational (p, p) Hodge class comes from an algebraic cycle?
- Precisely: Is the cycle class map

$$c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$$

surjective?
- True for $p = 1$: Lefschetz $(1, 1)$!
- very general abelian varieties (Recall, an element is *general* if it lies in the complement of finitely many proper closed subsets; very general if in the complement of countably many proper closed subsets)

Examples from moduli spaces

- Given $X :=$ very general smooth projective curve, r, d coprime integers, L line bundle on X of degree d. We have:
- Jacobian of X satisfies the Hodge conjecture.
- The moduli space of stable rank r, determinant L vector bundles on X satisfies the Hodge conjecture (rank 2: Balaji-King-Newstead, rank r: Biswas-Narasimhan).

Examples from moduli spaces

- Given $X :=$ very general smooth projective curve, r, d coprime integers, L line bundle on X of degree d. We have:
- Jacobian of X satisfies the Hodge conjecture.
- The moduli space of stable rank r, determinant L vector bundles on X satisfies the Hodge conjecture (rank 2: Balaji-King-Newstead, rank r: Biswas-Narasimhan).

Examples from moduli spaces

- Given $X :=$ very general smooth projective curve,
  r, d coprime integers,
  L line bundle on X of degree d. We have:
- Jacobian of X satisfies the Hodge conjecture.
- The moduli space of stable rank r, determinant L vector bundles on X satisfies the Hodge conjecture (rank 2: Balaji-King-Newstead, rank r: Biswas-Narasimhan).

Questions and Problems

- **Question**: For X a singular variety. Is there a formulation of the Hodge conjecture?
- There is a homological version by Jannsen (still unproven!).
- **Problem** : The classical Chow group is not compatible with arbitrary pull-back morphisms.
- Use the **Operational Chow group** by Fulton and Macpherson.
- **Question**: What about the cycle class map?
- One of the many issues: Hodge structure is no longer pure.

Questions and Problems

- **Question**: For X a singular variety. Is there a formulation of the Hodge conjecture?
- There is a homological version by Jannsen (still unproven!).
- **Problem** : The classical Chow group is not compatible with arbitrary pull-back morphisms.
- Use the **Operational Chow group** by Fulton and Macpherson.
- **Question**: What about the cycle class map?
- One of the many issues: Hodge structure is no longer pure.

Questions and Problems

- **Question**: For X a singular variety. Is there a formulation of the Hodge conjecture?
- There is a homological version by Jannsen (still unproven!).
- **Problem** : The classical Chow group is not compatible with arbitrary pull-back morphisms.
- Use the **Operational Chow group** by Fulton and Macpherson.
- **Question**: What about the cycle class map?
- One of the many issues: Hodge structure is no longer pure.

Questions and Problems

- **Question**: For X a singular variety. Is there a formulation of the Hodge conjecture?
- There is a homological version by Jannsen (still unproven!).
- **Problem** : The classical Chow group is not compatible with arbitrary pull-back morphisms.
- Use the **Operational Chow group** by Fulton and Macpherson.
- **Question**: What about the cycle class map?
- One of the many issues: Hodge structure is no longer pure.

Questions and Problems

- **Question**: For X a singular variety. Is there a formulation of the Hodge conjecture?
- There is a homological version by Jannsen (still unproven!).
- **Problem** : The classical Chow group is not compatible with arbitrary pull-back morphisms.
- Use the **Operational Chow group** by Fulton and Macpherson.
- **Question**: What about the cycle class map?
- One of the many issues: Hodge structure is no longer pure.

Questions and Problems

- **Question**: For X a singular variety. Is there a formulation of the Hodge conjecture?
- There is a homological version by Jannsen (still unproven!).
- **Problem** : The classical Chow group is not compatible with arbitrary pull-back morphisms.
- Use the **Operational Chow group** by Fulton and Macpherson.
- **Question**: What about the cycle class map?
- One of the many issues: Hodge structure is no longer pure.

Mixed Hodge structure

- A **mixed Hodge structure**, denoted $(V_{\mathbb{Z}}, W_{\bullet}, F^{\bullet})$ consists of a $\mathbb{Z}$ module $V_{\mathbb{Z}}$ with

  an increasing filtration $\dots W_0 \subseteq W_1 \subseteq W_2 \dots$ on $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$,

  a decreasing filtration: $V_{\mathbb{C}} = F^0 \supseteq F^1 \supseteq F^2 \dots$

  such that $F^{\bullet}$ defines a (pure) Hodge structure of weight k on the graded piece $Gr_k^W V_{\mathbb{Q}} = W_{k+1} V_{\mathbb{Q}} / W_k V_{\mathbb{Q}}$.

Mixed Hodge structure

- A **mixed Hodge structure**, denoted $(V_{\mathbb{Z}}, W_\bullet, F^\bullet)$ consists of a $\mathbb{Z}$ module $V_{\mathbb{Z}}$ with

  an increasing filtration $\dots W_0 \subseteq W_1 \subseteq W_2 \dots$ on $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$,

  a decreasing filtration: $V_{\mathbb{C}} = F^0 \supseteq F^1 \supseteq F^2 \dots$

  such that $F^\bullet$ defines a (pure) Hodge structure of weight k on the graded piece $Gr_k^W V_{\mathbb{Q}} = W_{k+1}V_{\mathbb{Q}}/W_k V_{\mathbb{Q}}$.

Mixed Hodge structure

- A **mixed Hodge structure**, denoted $(V_{\mathbb{Z}}, W_\bullet, F^\bullet)$ consists of a $\mathbb{Z}$ module $V_{\mathbb{Z}}$ with

  an increasing filtration $\ldots W_0 \subseteq W_1 \subseteq W_2 \ldots$ on $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$,

  a decreasing filtration: $V_{\mathbb{C}} = F^0 \supseteq F^1 \supseteq F^2 \ldots$

  such that $F^\bullet$ defines a (pure) Hodge structure of weight k on the graded piece $Gr_k^W V_{\mathbb{Q}} = W_{k+1} V_{\mathbb{Q}} / W_k V_{\mathbb{Q}}$.

Mixed Hodge structure

- A **mixed Hodge structure**, denoted $(V_{\mathbb{Z}}, W_\bullet, F^\bullet)$ consists of a $\mathbb{Z}$ module $V_{\mathbb{Z}}$ with

  an increasing filtration $\dots W_0 \subseteq W_1 \subseteq W_2 \dots$ on $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$,

  a decreasing filtration: $V_{\mathbb{C}} = F^0 \supseteq F^1 \supseteq F^2 \dots$

  such that $F^\bullet$ defines a (pure) Hodge structure of weight k on the graded piece $Gr_k^W V_{\mathbb{Q}} = W_{k+1} V_{\mathbb{Q}} / W_k V_{\mathbb{Q}}$.

Operational Chow group

- The operational Chow group $A^p(X)$ has several nice properties.
- It exists for singular as well as quasi-projective varieties.
- If X is non-singular, it is the classical Chow group i.e $Z^p(X)$ modulo rational equivalence.
- If X is non-singular, $\overline{X}$ a compactification of X with boundary $Z := \overline{X} \backslash X$, we have:

$$0 \to A^p_c(X) \to A^p(\overline{X}) \to A^p(Z),$$

where $A^p_c(X)$ is the *compactly supported operational Chow cohomology*

- If X is the union of two proper closed subvarieties X_1 and X_2, we have

$$0 \to A^p(X) \to A^p(X_1) \oplus A^p(X_2) \to A^p(X_1 \cap X_2)$$

Operational Chow group

- The operational Chow group $A^p(X)$ has several nice properties.
- It exists for singular as well as quasi-projective varieties.
- If X is non-singular, it is the classical Chow group i.e $Z^p(X)$ modulo rational equivalence.
- If X is non-singular, $\overline{X}$ a compactification of X with boundary $Z := \overline{X} \backslash X$, we have:

$$0 \to A^p_c(X) \to A^p(\overline{X}) \to A^p(Z),$$

  where $A^p_c(X)$ is the *compactly supported operational Chow cohomology*
- If X is the union of two proper closed subvarieties X_1 and X_2, we have

$$0 \to A^p(X) \to A^p(X_1) \oplus A^p(X_2) \to A^p(X_1 \cap X_2)$$

Operational Chow group

- The operational Chow group $A^p(X)$ has several nice properties.
- It exists for singular as well as quasi-projective varieties.
- If X is non-singular, it is the classical Chow group i.e $Z^p(X)$ modulo rational equivalence.
- If X is non-singular, $\overline{X}$ a compactification of X with boundary $Z := \overline{X} \backslash X$, we have:

$$0 \to A_c^p(X) \to A^p(\overline{X}) \to A^p(Z),$$

where $A_c^p(X)$ is the *compactly supported operational Chow cohomology*

- If X is the union of two proper closed subvarieties X_1 and X_2, we have

$$0 \to A^p(X) \to A^p(X_1) \oplus A^p(X_2) \to A^p(X_1 \cap X_2)$$

Operational Chow group

- The operational Chow group $A^p(X)$ has several nice properties.
- It exists for singular as well as quasi-projective varieties.
- If X is non-singular, it is the classical Chow group i.e $Z^p(X)$ modulo rational equivalence.
- If X is non-singular, $\overline{X}$ a compactification of X with boundary $Z := \overline{X} \backslash X$, we have:

$$0 \to A^p_c(X) \to A^p(\overline{X}) \to A^p(Z),$$

where $A^p_c(X)$ is the *compactly supported operational Chow cohomology*

- If X is the union of two proper closed subvarieties X_1 and X_2, we have

$$0 \to A^p(X) \to A^p(X_1) \oplus A^p(X_2) \to A^p(X_1 \cap X_2)$$

Operational Chow group

- The operational Chow group $A^p(X)$ has several nice properties.
- It exists for singular as well as quasi-projective varieties.
- If X is non-singular, it is the classical Chow group i.e $Z^p(X)$ modulo rational equivalence.
- If X is non-singular, $\overline{X}$ a compactification of X with boundary $Z := \overline{X} \backslash X$, we have:

$$0 \to A^p_c(X) \to A^p(\overline{X}) \to A^p(Z),$$

where $A^p_c(X)$ is the *compactly supported operational Chow cohomology*
- If X is the union of two proper closed subvarieties X_1 and X_2, we have

$$0 \to A^p(X) \to A^p(X_1) \oplus A^p(X_2) \to A^p(X_1 \cap X_2)$$

Bloch-Gille-Soulé cycle class map

- X a singular variety and $A^p(X)$ the Operational Chow group.
- Totaro: there is no map

$$A^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q})$$

with good properties.
- Bloch-Gillet-Soulé: there is a (functorial) cycle class map

$$\mathrm{cl}_p : A^p(X) \otimes \mathbb{Q} \to \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}).$$

- If X is non-singular, this agrees with the usual cycle class map.
- For X projective, define the *algebraic cohomology group* denoted by $H^{2p}_A(X) \subset \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q})$. to be the image of the cycle class map cl_p.

Bloch-Gille-Soulé cycle class map

- X a singular variety and $A^p(X)$ the Operational Chow group.
- Totaro: there is no map

$$A^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q})$$

with good properties.

- Bloch-Gillet-Soulé: there is a (functorial) cycle class map

$$\mathrm{cl}_p : A^p(X) \otimes \mathbb{Q} \to \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}).$$

- If X is non-singular, this agrees with the usual cycle class map.
- For X projective, define the *algebraic cohomology group* denoted by $H^{2p}_A(X) \subset \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q})$. to be the image of the cycle class map cl_p.

Bloch-Gille-Soulé cycle class map

- X a singular variety and $A^p(X)$ the Operational Chow group.
- Totaro: there is no map

$$A^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q})$$

with good properties.

- Bloch-Gillet-Soulé: there is a (functorial) cycle class map

$$\mathrm{cl}_p : A^p(X) \otimes \mathbb{Q} \to \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}).$$

- If X is non-singular, this agrees with the usual cycle class map.
- For X projective, define the *algebraic cohomology group* denoted by $H^{2p}_A(X) \subset \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q})$. to be the image of the cycle class map cl_p.

Bloch-Gille-Soulé cycle class map

- X a singular variety and $A^p(X)$ the Operational Chow group.
- Totaro: there is no map

$$A^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q})$$

with good properties.

- Bloch-Gillet-Soulé: there is a (functorial) cycle class map

$$\mathrm{cl}_p : A^p(X) \otimes \mathbb{Q} \to \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}).$$

- If X is non-singular, this agrees with the usual cycle class map.
- For X projective, define the *algebraic cohomology group* denoted by $H^{2p}_A(X) \subset \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q})$. to be the image of the cycle class map cl_p.

Bloch-Gille-Soulé cycle class map

- X a singular variety and $A^p(X)$ the Operational Chow group.
- Totaro: there is no map

$$A^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q})$$

with good properties.

- Bloch-Gillet-Soulé: there is a (functorial) cycle class map

$$\mathrm{cl}_p : A^p(X) \otimes \mathbb{Q} \to \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}).$$

- If X is non-singular, this agrees with the usual cycle class map.
- For X projective, define the *algebraic cohomology group* denoted by $H^{2p}_A(X) \subset \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q})$. to be the image of the cycle class map cl_p.

A cohomological formulation

- X projective variety of dimension n.
- Suppose singular locus of X, X_{sing} is of dimension at most $p-1$.
- Then we say that X satisfies the **Singular Hodge conjecture(SHC) in weight p** if the image of the cycle class map cl_p equals $H^{2p}_{\mathrm{Hdg}}(X) := \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}) \cap F^p H^{2p}(X, \mathbb{C})$.
- **Singular Hodge conjecture(SHC)**: The above statement is true for all p.

A cohomological formulation

- X projective variety of dimension n.
- Suppose singular locus of X, X_{sing} is of dimension at most $p-1$.
- Then we say that X satisfies the **Singular Hodge conjecture(SHC) in weight p** if the image of the cycle class map cl_p equals $H^{2p}_{\mathrm{Hdg}}(X) := \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}) \cap F^p H^{2p}(X, \mathbb{C})$.
- **Singular Hodge conjecture(SHC)**: The above statement is true for all p.

A cohomological formulation

- X projective variety of dimension n.
- Suppose singular locus of X, X_{sing} is of dimension at most $p-1$.
- Then we say that X satisfies the **Singular Hodge conjecture(SHC) in weight p** if the image of the cycle class map cl_p equals $H^{2p}_{\mathrm{Hdg}}(X) := \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}) \cap F^p H^{2p}(X, \mathbb{C})$.
- **Singular Hodge conjecture(SHC)**: The above statement is true for all p.

A cohomological formulation

- X projective variety of dimension n.
- Suppose singular locus of X, X_{sing} is of dimension at most $p-1$.
- Then we say that X satisfies the **Singular Hodge conjecture(SHC) in weight p** if the image of the cycle class map cl_p equals $H^{2p}_{\mathrm{Hdg}}(X) := \mathrm{Gr}^W_{2p} H^{2p}(X, \mathbb{Q}) \cap F^p H^{2p}(X, \mathbb{C})$.
- **Singular Hodge conjecture(SHC)**: The above statement is true for all p.

A cohomological formulation

- X projective variety of dimension n.
- Suppose singular locus of X, X_{sing} is of dimension at most $p-1$.
- Then we say that X satisfies the **Singular Hodge conjecture(SHC) in weight p** if the image of the cycle class map cl_p equals $H^{2p}_{\mathrm{Hdg}}(X) := \mathrm{Gr}^W_{2p} H^{2p}(X,\mathbb{Q}) \cap F^p H^{2p}(X,\mathbb{C})$.
- **Singular Hodge conjecture(SHC)**: The above statement is true for all p.

Goal

- For X non-singular, SHC is just the classical Hodge conjecture.
- **Goal**: Give a criterion for a singular variety to satisfy the SHC.
- **Strategy**(Heuristic):
  - Embed the singular variety in a flat family of projective varieties such that the given variety is the singular fibre.
  - Equip the singular fibre with the limit mixed Hodge structure due to Schmid.
  - Ask, if a very general fibre in the family satisfy the Hodge conjecture, does the central fibre satisfy SHC?

Goal

- For X non-singular, SHC is just the classical Hodge conjecture.
- **Goal**: Give a criterion for a singular variety to satisfy the SHC.
- **Strategy**(Heuristic):
  - Embed the singular variety in a flat family of projective varieties such that the given variety is the singular fibre.
  - Equip the singular fibre with the limit mixed Hodge structure due to Schmid.
  - Ask, if a very general fibre in the family satisfy the Hodge conjecture, does the central fibre satisfy SHC?

Goal

- For X non-singular, SHC is just the classical Hodge conjecture.
- **Goal**: Give a criterion for a singular variety to satisfy the SHC.
- **Strategy**(Heuristic):
  - Embed the singular variety in a flat family of projective varieties such that the given variety is the singular fibre.
  - Equip the singular fibre with the limit mixed Hodge structure due to Schmid.
  - Ask, if a very general fibre in the family satisfy the Hodge conjecture, does the central fibre satisfy SHC?

Goal

- For X non-singular, SHC is just the classical Hodge conjecture.
- **Goal**: Give a criterion for a singular variety to satisfy the SHC.
- **Strategy**(Heuristic):
    - Embed the singular variety in a flat family of projective varieties such that the given variety is the singular fibre.
    - Equip the singular fibre with the limit mixed Hodge structure due to Schmid.
    - Ask, if a very general fibre in the family satisfy the Hodge conjecture, does the central fibre satisfy SHC?

Goal

- For X non-singular, SHC is just the classical Hodge conjecture.
- **Goal**: Give a criterion for a singular variety to satisfy the SHC.
- **Strategy**(Heuristic):
  - Embed the singular variety in a flat family of projective varieties such that the given variety is the singular fibre.
  - Equip the singular fibre with the limit mixed Hodge structure due to Schmid.
  - Ask, if a very general fibre in the family satisfy the Hodge conjecture, does the central fibre satisfy SHC?

Goal

- For X non-singular, SHC is just the classical Hodge conjecture.
- **Goal**: Give a criterion for a singular variety to satisfy the SHC.
- **Strategy**(Heuristic):
  - Embed the singular variety in a flat family of projective varieties such that the given variety is the singular fibre.
  - Equip the singular fibre with the limit mixed Hodge structure due to Schmid.
  - Ask, if a very general fibre in the family satisfy the Hodge conjecture, does the central fibre satisfy SHC?

Limit mixed Hodge structure

- Let $\pi : \mathcal{X} \to \Delta$ flat family of projective varieties, smooth over Δ^* with $\pi^{-1}(0) := \mathcal{X}_0 = X$.
- Let $e : \mathfrak{h} \to \Delta^*$ be the universal covering.
- Denote by $\mathcal{X}_\infty$ the pull-back of $\mathcal{X}$ to $\mathfrak{h}$.
- By Ehresmann's theorem, given any $s \in \mathfrak{h}$, there is a canonical identification between $H^{2p}(\mathcal{X}_\infty, \mathbb{Z})$ and $H^{2p}(\mathcal{X}_s, \mathbb{Z})$.
- The Hodge filtration on $H^{2p}(\mathcal{X}_s, \mathbb{C})$ induces a Hodge filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$, say $F_s^\bullet$.
- The limit "Hodge" filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$ is the limit of $F_s^\bullet$ (twisted by the monodromy action) as the imaginary part of s tends to ∞.

Limit mixed Hodge structure

- Let $\pi : \mathcal{X} \to \Delta$ flat family of projective varieties, smooth over Δ^* with $\pi^{-1}(0) := \mathcal{X}_0 = X$.
- Let $e : \mathfrak{h} \to \Delta^*$ be the universal covering.
- Denote by $\mathcal{X}_\infty$ the pull-back of $\mathcal{X}$ to $\mathfrak{h}$.
- By Ehresmann's theorem, given any $s \in \mathfrak{h}$, there is a canonical identification between $H^{2p}(\mathcal{X}_\infty, \mathbb{Z})$ and $H^{2p}(\mathcal{X}_s, \mathbb{Z})$.
- The Hodge filtration on $H^{2p}(\mathcal{X}_s, \mathbb{C})$ induces a Hodge filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$, say $F_s^\bullet$.
- The limit "Hodge" filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$ is the limit of $F_s^\bullet$ (twisted by the monodromy action) as the imaginary part of s tends to ∞.

Limit mixed Hodge structure

- Let $\pi : \mathcal{X} \to \Delta$ flat family of projective varieties, smooth over Δ^* with $\pi^{-1}(0) := \mathcal{X}_0 = X$.
- Let $e : \mathfrak{h} \to \Delta^*$ be the universal covering.
- Denote by $\mathcal{X}_\infty$ the pull-back of $\mathcal{X}$ to $\mathfrak{h}$.
- By Ehresmann's theorem, given any $s \in \mathfrak{h}$, there is a canonical identification between $H^{2p}(\mathcal{X}_\infty, \mathbb{Z})$ and $H^{2p}(\mathcal{X}_s, \mathbb{Z})$.
- The Hodge filtration on $H^{2p}(\mathcal{X}_s, \mathbb{C})$ induces a Hodge filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$, say $F_s^\bullet$.
- The limit "Hodge" filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$ is the limit of $F_s^\bullet$ (twisted by the monodromy action) as the imaginary part of s tends to ∞.

Limit mixed Hodge structure

- Let $\pi : \mathcal{X} \to \Delta$ flat family of projective varieties, smooth over Δ^* with $\pi^{-1}(0) := \mathcal{X}_0 = X$.
- Let $e : \mathfrak{h} \to \Delta^*$ be the universal covering.
- Denote by $\mathcal{X}_\infty$ the pull-back of $\mathcal{X}$ to $\mathfrak{h}$.
- By Ehresmann's theorem, given any $s \in \mathfrak{h}$, there is a canonical identification between $H^{2p}(\mathcal{X}_\infty, \mathbb{Z})$ and $H^{2p}(\mathcal{X}_s, \mathbb{Z})$.
- The Hodge filtration on $H^{2p}(\mathcal{X}_s, \mathbb{C})$ induces a Hodge filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$, say $F_s^\bullet$.
- The limit "Hodge" filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$ is the limit of $F_s^\bullet$ (twisted by the monodromy action) as the imaginary part of s tends to ∞.

Limit mixed Hodge structure

- Let $\pi : \mathcal{X} \to \Delta$ flat family of projective varieties, smooth over Δ^* with $\pi^{-1}(0) := \mathcal{X}_0 = X$.
- Let $e : \mathfrak{h} \to \Delta^*$ be the universal covering.
- Denote by $\mathcal{X}_\infty$ the pull-back of $\mathcal{X}$ to $\mathfrak{h}$.
- By Ehresmann's theorem, given any $s \in \mathfrak{h}$, there is a canonical identification between $H^{2p}(\mathcal{X}_\infty, \mathbb{Z})$ and $H^{2p}(\mathcal{X}_s, \mathbb{Z})$.
- The Hodge filtration on $H^{2p}(\mathcal{X}_s, \mathbb{C})$ induces a Hodge filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$, say $F_s^\bullet$.
- The limit "Hodge" filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$ is the limit of $F_s^\bullet$ (twisted by the monodromy action) as the imaginary part of s tends to ∞.

Limit mixed Hodge structure

- Let $\pi : \mathcal{X} \to \Delta$ flat family of projective varieties, smooth over Δ^* with $\pi^{-1}(0) := \mathcal{X}_0 = X$.
- Let $e : \mathfrak{h} \to \Delta^*$ be the universal covering.
- Denote by $\mathcal{X}_\infty$ the pull-back of $\mathcal{X}$ to $\mathfrak{h}$.
- By Ehresmann's theorem, given any $s \in \mathfrak{h}$, there is a canonical identification between $H^{2p}(\mathcal{X}_\infty, \mathbb{Z})$ and $H^{2p}(\mathcal{X}_s, \mathbb{Z})$.
- The Hodge filtration on $H^{2p}(\mathcal{X}_s, \mathbb{C})$ induces a Hodge filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$, say $F_s^\bullet$.
- The limit "Hodge" filtration on $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$ is the limit of $F_s^\bullet$ (twisted by the monodromy action) as the imaginary part of s tends to ∞.

Can we use the specialization map...

- Use the specialization morphism to relate the mixed Hodge structure on the cohomology of the singular fibre $H^{2p}(X, \mathbb{C})$ to that of a general fibre equipped with the limit mixed Hodge structure $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$.
- In general, the specialization morphism is neither surjective nor injective.
- We show that for the algebraic classes, it is surjective and study the kernel.
- But this is not enough!

Can we use the specialization map...

- Use the specialization morphism to relate the mixed Hodge structure on the cohomology of the singular fibre $H^{2p}(X, \mathbb{C})$ to that of a general fibre equipped with the limit mixed Hodge structure $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$.
- In general, the specialization morphism is neither surjective nor injective.
- We show that for the algebraic classes, it is surjective and study the kernel.
- But this is not enough!

Can we use the specialization map...

- Use the specialization morphism to relate the mixed Hodge structure on the cohomology of the singular fibre $H^{2p}(X, \mathbb{C})$ to that of a general fibre equipped with the limit mixed Hodge structure $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$.
- In general, the specialization morphism is neither surjective nor injective.
- We show that for the algebraic classes, it is surjective and study the kernel.
- But this is not enough!

Can we use the specialization map...

- Use the specialization morphism to relate the mixed Hodge structure on the cohomology of the singular fibre $H^{2p}(X, \mathbb{C})$ to that of a general fibre equipped with the limit mixed Hodge structure $H^{2p}(\mathcal{X}_\infty, \mathbb{C})$.
- In general, the specialization morphism is neither surjective nor injective.
- We show that for the algebraic classes, it is surjective and study the kernel.
- But this is not enough!

Technical problems: Mumford-Tate families

- **Problem**: The rank of the Hodge lattice of the limit mixed Hodge structure could be larger than that of the Hodge lattice of the general fibre.
- If so, we cannot hope to intercept all the Hodge classes of the singular fibre from those of the smooth fibre.
- In other words, we need that the rank of the Hodge lattice of the limit mixed Hodge structure is equal to the rank of the Hodge lattice of a general fibre.
- Such a family $\pi : \mathcal{X} \to \Delta$ will be called a *Mumford-Tate family*.

Technical problems: Mumford-Tate families

- **Problem**: The rank of the Hodge lattice of the limit mixed Hodge structure could be larger than that of the Hodge lattice of the general fibre.
- If so, we cannot hope to intercept all the Hodge classes of the singular fibre from those of the smooth fibre.
- In other words, we need that the rank of the Hodge lattice of the limit mixed Hodge structure is equal to the rank of the Hodge lattice of a general fibre.
- Such a family $\pi : \mathcal{X} \to \Delta$ will be called a *Mumford-Tate family*.

Technical problems: Mumford-Tate families

- **Problem**: The rank of the Hodge lattice of the limit mixed Hodge structure could be larger than that of the Hodge lattice of the general fibre.
- If so, we cannot hope to intercept all the Hodge classes of the singular fibre from those of the smooth fibre.
- In other words, we need that the rank of the Hodge lattice of the limit mixed Hodge structure is equal to the rank of the Hodge lattice of a general fibre.
- Such a family $\pi : \mathcal{X} \to \Delta$ will be called a *Mumford-Tate family*.

Technical problems: Mumford-Tate families

- **Problem**: The rank of the Hodge lattice of the limit mixed Hodge structure could be larger than that of the Hodge lattice of the general fibre.
- If so, we cannot hope to intercept all the Hodge classes of the singular fibre from those of the smooth fibre.
- In other words, we need that the rank of the Hodge lattice of the limit mixed Hodge structure is equal to the rank of the Hodge lattice of a general fibre.
- Such a family $\pi : \mathcal{X} \to \Delta$ will be called a *Mumford-Tate family*.

Alternate definition of a Hodge structure

- Denote by $\mathbb{S}$ the Weil restriction of scalars for the field extension $\mathbb{C}/\mathbb{R}$.
- For a commutative $\mathbb{R}$-algebra A, $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$
- $\mathbb{S}$ is an $\mathbb{R}$-algebraic group.
- **Definition**: A Hodge structure of weight r is given by a non-constant homomorphism of $\mathbb{R}$-algebraic groups.

$$\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

such that over $\mathbb{C}$ the eigenspace decomposition of $\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{C}})$ is:
  - $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$
  - $\phi(z)v = z^p \overline{z}^q v, \quad v \in V^{p,q}$

Alternate definition of a Hodge structure

- Denote by $\mathbb{S}$ the Weil restriction of scalars for the field extension $\mathbb{C}/\mathbb{R}$.
- For a commutative $\mathbb{R}$-algebra A, $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$
- $\mathbb{S}$ is an $\mathbb{R}$-algebraic group.
- **Definition**: A Hodge structure of weight r is given by a non-constant homomorphism of $\mathbb{R}$-algebraic groups.

$$\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

  such that over $\mathbb{C}$ the eigenspace decomposition of $\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{C}})$ is:
  - $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$
  - $\phi(z)v = z^p \overline{z}^q v, \quad v \in V^{p,q}$

Alternate definition of a Hodge structure

- Denote by $\mathbb{S}$ the Weil restriction of scalars for the field extension $\mathbb{C}/\mathbb{R}$.
- For a commutative $\mathbb{R}$-algebra A, $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$
- $\mathbb{S}$ is an $\mathbb{R}$-algebraic group.
- **Definition**: A Hodge structure of weight r is given by a non-constant homomorphism of $\mathbb{R}$-algebraic groups.

$$\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

such that over $\mathbb{C}$ the eigenspace decomposition of $\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{C}})$ is:

  - $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$
  - $\phi(z)v = z^p \overline{z}^q v, \quad v \in V^{p,q}$

Alternate definition of a Hodge structure

- Denote by $\mathbb{S}$ the Weil restriction of scalars for the field extension $\mathbb{C}/\mathbb{R}$.
- For a commutative $\mathbb{R}$-algebra A, $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$
- $\mathbb{S}$ is an $\mathbb{R}$-algebraic group.
- **Definition**: A Hodge structure of weight r is given by a non-constant homomorphism of $\mathbb{R}$-algebraic groups.

$$\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

such that over $\mathbb{C}$ the eigenspace decomposition of $\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{C}})$ is:

- $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$
- $\phi(z)v = z^p \overline{z}^q v, \quad v \in V^{p,q}$

Alternate definition of a Hodge structure

- Denote by $\mathbb{S}$ the Weil restriction of scalars for the field extension $\mathbb{C}/\mathbb{R}$.
- For a commutative $\mathbb{R}$-algebra A, $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$
- $\mathbb{S}$ is an $\mathbb{R}$-algebraic group.
- **Definition**: A Hodge structure of weight r is given by a non-constant homomorphism of $\mathbb{R}$-algebraic groups.

$$\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

such that over $\mathbb{C}$ the eigenspace decomposition of $\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{C}})$ is:

- $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$
- $\phi(z)v = z^p \overline{z}^q v, \quad v \in V^{p,q}$

Alternate definition of a Hodge structure

- Denote by $\mathbb{S}$ the Weil restriction of scalars for the field extension $\mathbb{C}/\mathbb{R}$.
- For a commutative $\mathbb{R}$-algebra A, $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$
- $\mathbb{S}$ is an $\mathbb{R}$-algebraic group.
- **Definition**: A Hodge structure of weight r is given by a non-constant homomorphism of $\mathbb{R}$-algebraic groups.

$$\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

  such that over $\mathbb{C}$ the eigenspace decomposition of $\phi : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{C}})$ is:
  - $V_{\mathbb{C}} = \oplus_{p+q=r} V^{p,q}$
  - $\phi(z)v = z^p \overline{z}^q v, \quad v \in V^{p,q}$

Definition of Mumford-Tate group

- The **Mumford-Tate** group associated to a Hodge structure (V, ϕ) is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ containing the image $\phi(\mathbb{S})$.
- Denote by $T^{m,n}(V) := V^{\otimes m} \otimes (\mathrm{Hom}(V, \mathbb{Z}))^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap T^{m,n}(V_{\mathbb{Q}})$ of weight 0 are called **Hodge tensors**. They inherit a natural Hodge structure from V.
- When V is equipped with a mixed Hodge structure, these are weight 0 elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap W_0 T^{m,n}(V_{\mathbb{Q}})$.
- Alternate definition of Mumford-Tate group: The Mumford-Tate group is the biggest subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which fixes the Hodge tensors.

Definition of Mumford-Tate group

- The **Mumford-Tate** group associated to a Hodge structure (V, ϕ) is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ containing the image $\phi(\mathbb{S})$.
- Denote by $T^{m,n}(V) := V^{\otimes m} \otimes (\mathrm{Hom}(V, \mathbb{Z}))^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap T^{m,n}(V_{\mathbb{Q}})$ of weight 0 are called **Hodge tensors**. They inherit a natural Hodge structure from V.
- When V is equipped with a mixed Hodge structure, these are weight 0 elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap W_0 T^{m,n}(V_{\mathbb{Q}})$.
- Alternate definition of Mumford-Tate group: The Mumford-Tate group is the biggest subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which fixes the Hodge tensors.

Definition of Mumford-Tate group

- The **Mumford-Tate** group associated to a Hodge structure (V, ϕ) is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ containing the image $\phi(\mathbb{S})$.
- Denote by $T^{m,n}(V) := V^{\otimes m} \otimes (\mathrm{Hom}(V, \mathbb{Z}))^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap T^{m,n}(V_{\mathbb{Q}})$ of weight 0 are called **Hodge tensors**. They inherit a natural Hodge structure from V.
- When V is equipped with a mixed Hodge structure, these are weight 0 elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap W_0 T^{m,n}(V_{\mathbb{Q}})$.
- Alternate definition of Mumford-Tate group: The Mumford-Tate group is the biggest subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which fixes the Hodge tensors.

Definition of Mumford-Tate group

- The **Mumford-Tate** group associated to a Hodge structure (V, ϕ) is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ containing the image $\phi(\mathbb{S})$.
- Denote by $T^{m,n}(V) := V^{\otimes m} \otimes (\mathrm{Hom}(V, \mathbb{Z}))^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap T^{m,n}(V_{\mathbb{Q}})$ of weight 0 are called **Hodge tensors**. They inherit a natural Hodge structure from V.
- When V is equipped with a mixed Hodge structure, these are weight 0 elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap W_0 T^{m,n}(V_{\mathbb{Q}})$.
- Alternate definition of Mumford-Tate group: The Mumford-Tate group is the biggest subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which fixes the Hodge tensors.

Definition of Mumford-Tate group

- The **Mumford-Tate** group associated to a Hodge structure (V,ϕ) is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ containing the image $\phi(\mathbb{S})$.
- Denote by $T^{m,n}(V) := V^{\otimes m} \otimes (\mathrm{Hom}(V,\mathbb{Z}))^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap T^{m,n}(V_{\mathbb{Q}})$ of weight 0 are called **Hodge tensors**. They inherit a natural Hodge structure from V.
- When V is equipped with a mixed Hodge structure, these are weight 0 elements of $F^0(T^{m,n}(V_{\mathbb{C}})) \cap W_0 T^{m,n}(V_{\mathbb{Q}})$.
- Alternate definition of Mumford-Tate group: The Mumford-Tate group is the biggest subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ which fixes the Hodge tensors.

Mumford-Tate family

- Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over Δ^*.
- Fix $p > 0$.
- For $t \in \Delta^*$, denote by M_t the Mumford-Tate group associated to the pure Hodge structure on $H^{2p}(\mathcal{X}_t, \mathbb{Q})$.
- Recall, $H^{2p}(\mathcal{X}_t, \mathbb{Q})$ can also be equipped with limit mixed Hodge structure (l.m.h.s).
- Denote by M_t^∞ the Mumford-Tate group associated to the l.m.h.s on $H^{2p}(\mathcal{X}_\infty, \mathbb{Q})$.
- **Definition**: We call π a *Mumford-Tate family* if for a general $t \in \Delta^*$,

$$M_t \cong M_t^\infty$$

are isomorphic as algebraic subgroups of $\mathrm{Aut}(H^{2p}(X_t, \mathbb{Q}))$.
- **Example** Relative Gieseker Moduli space of stable sheaves of rank 2 and odd degree determinant.

Mumford-Tate family

- Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over Δ^*.
- Fix $p > 0$.
- For $t \in \Delta^*$, denote by M_t the Mumford-Tate group associated to the pure Hodge structure on $H^{2p}(\mathcal{X}_t, \mathbb{Q})$.
- Recall, $H^{2p}(\mathcal{X}_t, \mathbb{Q})$ can also be equipped with limit mixed Hodge structure (l.m.h.s).
- Denote by M_t^∞ the Mumford-Tate group associated to the l.m.h.s on $H^{2p}(\mathcal{X}_\infty, \mathbb{Q})$.
- **Definition**: We call π a *Mumford-Tate family* if for a general $t \in \Delta^*$,

$$M_t \cong M_t^\infty$$

are isomorphic as algebraic subgroups of $\mathrm{Aut}(H^{2p}(X_t, \mathbb{Q}))$.
- **Example** Relative Gieseker Moduli space of stable sheaves of rank 2 and odd degree determinant.

Mumford-Tate family

- Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over Δ^*.
- Fix $p > 0$.
- For $t \in \Delta^*$, denote by M_t the Mumford-Tate group associated to the pure Hodge structure on $H^{2p}(\mathcal{X}_t, \mathbb{Q})$.
- Recall, $H^{2p}(\mathcal{X}_t, \mathbb{Q})$ can also be equipped with limit mixed Hodge structure (l.m.h.s).
- Denote by M_t^∞ the Mumford-Tate group associated to the l.m.h.s on $H^{2p}(\mathcal{X}_\infty, \mathbb{Q})$.
- **Definition**: We call π a *Mumford-Tate family* if for a general $t \in \Delta^*$,

$$M_t \cong M_t^\infty$$

  are isomorphic as algebraic subgroups of $\mathrm{Aut}(H^{2p}(X_t, \mathbb{Q}))$.
- **Example** Relative Gieseker Moduli space of stable sheaves of rank 2 and odd degree determinant.

Mumford-Tate family

- Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over Δ^*.
- Fix $p > 0$.
- For $t \in \Delta^*$, denote by M_t the Mumford-Tate group associated to the pure Hodge structure on $H^{2p}(\mathcal{X}_t, \mathbb{Q})$.
- Recall, $H^{2p}(\mathcal{X}_t, \mathbb{Q})$ can also be equipped with limit mixed Hodge structure (l.m.h.s).
- Denote by M_t^∞ the Mumford-Tate group associated to the l.m.h.s on $H^{2p}(\mathcal{X}_\infty, \mathbb{Q})$.
- **Definition**: We call π a *Mumford-Tate family* if for a general $t \in \Delta^*$,

$$M_t \cong M_t^\infty$$

are isomorphic as algebraic subgroups of $\mathrm{Aut}(H^{2p}(X_t, \mathbb{Q}))$.
- **Example** Relative Gieseker Moduli space of stable sheaves of rank 2 and odd degree determinant.

Mumford-Tate family

- Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over Δ^*.
- Fix $p > 0$.
- For $t \in \Delta^*$, denote by M_t the Mumford-Tate group associated to the pure Hodge structure on $H^{2p}(\mathcal{X}_t, \mathbb{Q})$.
- Recall, $H^{2p}(\mathcal{X}_t, \mathbb{Q})$ can also be equipped with limit mixed Hodge structure (l.m.h.s).
- Denote by M_t^∞ the Mumford-Tate group associated to the l.m.h.s on $H^{2p}(\mathcal{X}_\infty, \mathbb{Q})$.
- **Definition**: We call π a *Mumford-Tate family* if for a general $t \in \Delta^*$,

$$M_t \cong M_t^\infty$$

are isomorphic as algebraic subgroups of $\mathrm{Aut}(H^{2p}(X_t, \mathbb{Q}))$.
- **Example** Relative Gieseker Moduli space of stable sheaves of rank 2 and odd degree determinant.

Mumford-Tate family

- Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over Δ^*.
- Fix $p > 0$.
- For $t \in \Delta^*$, denote by M_t the Mumford-Tate group associated to the pure Hodge structure on $H^{2p}(\mathcal{X}_t, \mathbb{Q})$.
- Recall, $H^{2p}(\mathcal{X}_t, \mathbb{Q})$ can also be equipped with limit mixed Hodge structure (l.m.h.s).
- Denote by M_t^{∞} the Mumford-Tate group associated to the l.m.h.s on $H^{2p}(\mathcal{X}_{\infty}, \mathbb{Q})$.
- **Definition**: We call π a *Mumford-Tate family* if for a general $t \in \Delta^*$,
$$M_t \cong M_t^{\infty}$$
are isomorphic as algebraic subgroups of $\mathrm{Aut}(H^{2p}(X_t, \mathbb{Q}))$.
- **Example** Relative Gieseker Moduli space of stable sheaves of rank 2 and odd degree determinant.

Mumford-Tate family

- Let $\pi : \mathcal{X} \to \Delta$ be a smooth family over Δ^*.
- Fix $p > 0$.
- For $t \in \Delta^*$, denote by M_t the Mumford-Tate group associated to the pure Hodge structure on $H^{2p}(\mathcal{X}_t, \mathbb{Q})$.
- Recall, $H^{2p}(\mathcal{X}_t, \mathbb{Q})$ can also be equipped with limit mixed Hodge structure (l.m.h.s).
- Denote by M_t^∞ the Mumford-Tate group associated to the l.m.h.s on $H^{2p}(\mathcal{X}_\infty, \mathbb{Q})$.
- **Definition**: We call π a *Mumford-Tate family* if for a general $t \in \Delta^*$,
$$M_t \cong M_t^\infty$$
are isomorphic as algebraic subgroups of $\mathrm{Aut}(H^{2p}(X_t, \mathbb{Q}))$.
- **Example** Relative Gieseker Moduli space of stable sheaves of rank 2 and odd degree determinant.

Main Result

- X projective variety (possibly singular) of dimension n.
- Fix integer p such that the dimension of singular locus of X strictly less than p.
- **Theorem**(Dan, -): Let $\pi_0 : \mathcal{X} \to \Delta$ be a flat, projective Mumford-Tate family with $X = \mathcal{X}_0$.

  Suppose a general fibre over Δ^* satisfies the Hodge conjecture in weight p. If the Hodge conjecture in weight $p-1$ holds for varieties of dimension $n-1$, then X satisfies the singular Hodge conjecture in weight p.

Main Result

- X projective variety (possibly singular) of dimension n.
- Fix integer p such that the dimension of singular locus of X strictly less than p.
- **Theorem**(Dan, -): Let $\pi_0 : \mathcal{X} \to \Delta$ be a flat, projective Mumford-Tate family with $X = \mathcal{X}_0$.

  Suppose a general fibre over Δ^* satisfies the Hodge conjecture in weight p. If the Hodge conjecture in weight $p-1$ holds for varieties of dimension $n-1$, then X satisfies the singular Hodge conjecture in weight p.

Main Result

- X projective variety (possibly singular) of dimension n.
- Fix integer p such that the dimension of singular locus of X strictly less than p.
- **Theorem**(Dan, -): Let $\pi_0 : \mathcal{X} \to \Delta$ be a flat, projective Mumford-Tate family with $X = \mathcal{X}_0$.

  Suppose a general fibre over Δ^* satisfies the Hodge conjecture in weight p. If the Hodge conjecture in weight $p-1$ holds for varieties of dimension $n-1$, then X satisfies the singular Hodge conjecture in weight p.

Idea of the proof

- By semi-stable reduction, there exists a flat, projective family $\pi : \mathcal{Y} \to \Delta$ which has the same fibre over Δ^* as π_0, $\mathcal{Y}$ is regular and the central fibre is a reduced simple normal crossings divisor with one of the irreducible components, say $\widetilde{Y}$ being proper birational to X.
- Denote by $\mathcal{Y}_0$ the central fibre of π and E the union of the components of $\mathcal{Y}_0$, except for $\widetilde{Y}$ i.e.,
- $\mathcal{Y}_0 = \widetilde{Y} \cup E$
- Since Y_{sing} is of dimension at most $p-1$, $H^i(Y_{\mathrm{sing}}) = 0$ for $i \geq 2p-1$.

Idea of the proof

- By semi-stable reduction, there exists a flat, projective family $\pi : \mathcal{Y} \to \Delta$ which has the same fibre over Δ^* as π_0, $\mathcal{Y}$ is regular and the central fibre is a reduced simple normal crossings divisor with one of the irreducible components, say $\widetilde{Y}$ being proper birational to X.
- Denote by $\mathcal{Y}_0$ the central fibre of π and E the union of the components of $\mathcal{Y}_0$, except for $\widetilde{Y}$ i.e.,
- $\mathcal{Y}_0 = \widetilde{Y} \cup E$
- Since Y_{sing} is of dimension at most $p-1$, $H^i(Y_{\mathrm{sing}}) = 0$ for $i \geq 2p-1$.

Idea of the proof

- By semi-stable reduction, there exists a flat, projective family $\pi : \mathcal{Y} \to \Delta$ which has the same fibre over Δ^* as π_0, $\mathcal{Y}$ is regular and the central fibre is a reduced simple normal crossings divisor with one of the irreducible components, say $\widetilde{Y}$ being proper birational to X.
- Denote by $\mathcal{Y}_0$ the central fibre of π and E the union of the components of $\mathcal{Y}_0$, except for $\widetilde{Y}$ i.e.,
- $\mathcal{Y}_0 = \widetilde{Y} \cup E$
- Since Y_{sing} is of dimension at most $p-1$, $H^i(Y_{\mathrm{sing}}) = 0$ for $i \geq 2p-1$.

Idea of the proof

- By semi-stable reduction, there exists a flat, projective family $\pi : \mathcal{Y} \to \Delta$ which has the same fibre over Δ^* as π_0, $\mathcal{Y}$ is regular and the central fibre is a reduced simple normal crossings divisor with one of the irreducible components, say $\widetilde{Y}$ being proper birational to X.
- Denote by $\mathcal{Y}_0$ the central fibre of π and E the union of the components of $\mathcal{Y}_0$, except for $\widetilde{Y}$ i.e.,
- $\mathcal{Y}_0 = \widetilde{Y} \cup E$
- Since Y_{sing} is of dimension at most $p-1$, $H^i(Y_{\mathrm{sing}}) = 0$ for $i \geq 2p-1$.

Idea of proof ...

- Let U be the regular locus of X.
- Observe that $\mathrm{Gr}^W_{2p} H^{2p}_c(U) \cong \mathrm{Gr}^W_{2p} H^{2p}(X)$
- We have the following exact sequence of pure Hodge structures:

$$0 \to \mathrm{Gr}^W_{2p} H^{2p}_c(U, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(\mathcal{Y}_0, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(E, \mathbb{Q}).$$

- This gives rise to the exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \qquad (1)$$

- Similarly get an exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) \to H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y}). \qquad (2)$$

Idea of proof ...

- Let U be the regular locus of X.
- Observe that $\mathrm{Gr}^W_{2p} H^{2p}_c(U) \cong \mathrm{Gr}^W_{2p} H^{2p}(X)$
- We have the following exact sequence of pure Hodge structures:

$$0 \to \mathrm{Gr}^W_{2p} H^{2p}_c(U, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(\mathcal{Y}_0, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(E, \mathbb{Q}).$$

- This gives rise to the exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \qquad (1)$$

- Similarly get an exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) \to H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y}). \qquad (2)$$

Idea of proof ...

- Let U be the regular locus of X.
- Observe that $\mathrm{Gr}^W_{2p} H^{2p}_c(U) \cong \mathrm{Gr}^W_{2p} H^{2p}(X)$
- We have the following exact sequence of pure Hodge structures:

$$0 \to \mathrm{Gr}^W_{2p} H^{2p}_c(U, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(\mathcal{Y}_0, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(E, \mathbb{Q}).$$

- This gives rise to the exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \quad (1)$$

- Similarly get an exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) \to H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y}). \quad (2)$$

Idea of proof ...

- Let U be the regular locus of X.
- Observe that $\mathrm{Gr}^W_{2p} H^{2p}_c(U) \cong \mathrm{Gr}^W_{2p} H^{2p}(X)$
- We have the following exact sequence of pure Hodge structures:

$$0 \to \mathrm{Gr}^W_{2p} H^{2p}_c(U, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(\mathcal{Y}_0, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(E, \mathbb{Q}).$$

- This gives rise to the exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \qquad (1)$$

- Similarly get an exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) \to H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y}). \qquad (2)$$

Idea of proof ...

- Let U be the regular locus of X.
- Observe that $\mathrm{Gr}^W_{2p} H^{2p}_c(U) \cong \mathrm{Gr}^W_{2p} H^{2p}(X)$
- We have the following exact sequence of pure Hodge structures:

$$0 \to \mathrm{Gr}^W_{2p} H^{2p}_c(U, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(\mathcal{Y}_0, \mathbb{Q}) \to \mathrm{Gr}^W_{2p} H^{2p}(E, \mathbb{Q}).$$

- This gives rise to the exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \quad (1)$$

- Similarly get an exact sequence (of $\mathbb{Q}$-vector spaces):

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) \to H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y}). \quad (2)$$

Idea of proof ...

- By properties of the operational Chow group:

$$0 \to A^p(X) \to A^p(\mathcal{Y}_0) \to A^p(E) \tag{3}$$

$$0 \to A^p(X) \to A^p(\widetilde{Y}) \to A^p(E \cap \widetilde{Y}) \tag{4}$$

- The cycle class map cl_p induces a morphism of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^p(X) & \longrightarrow & A^p(\mathcal{Y}_0) & \longrightarrow & A^p(E) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(X) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(E) \end{array}$$

Idea of proof ...

- By properties of the operational Chow group:

$$0 \to A^p(X) \to A^p(\mathcal{Y}_0) \to A^p(E) \quad (3)$$

$$0 \to A^p(X) \to A^p(\widetilde{Y}) \to A^p(E \cap \widetilde{Y}) \quad (4)$$

- The cycle class map cl_p induces a morphism of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^p(X) & \longrightarrow & A^p(\mathcal{Y}_0) & \longrightarrow & A^p(E) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(X) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(E) \end{array}$$

Idea of proof ...

- and

$$\begin{array}{ccccccc}
0 & \longrightarrow & A^p(X) & \longrightarrow & A^p(\widetilde{Y}) & \longrightarrow & A^p(E \cap \widetilde{Y}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(X) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y})
\end{array}$$

- Suffices to show: the cycle class map from $A^p(\mathcal{Y}_0)$ to $H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0)$ is surjective i.e., $H^{2p}_A(\mathcal{Y}_0) = H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0)$.
- Since

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \qquad (5)$$

is an exact sequence of $\mathbb{Q}$-vector spaces, this implies the cycle class map from $A^p(X)$ to $H^{2p}_{\mathrm{Hdg}}(X)$ is surjective.

Idea of proof ...

- and

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^p(X) & \longrightarrow & A^p(\widetilde{Y}) & \longrightarrow & A^p(E \cap \widetilde{Y}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(X) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y}) \end{array}$$

- Suffices to show: the cycle class map from $A^p(\mathcal{Y}_0)$ to $H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0)$ is surjective i.e., $H^{2p}_A(\mathcal{Y}_0) = H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0)$.
- Since

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \tag{5}$$

is an exact sequence of $\mathbb{Q}$-vector spaces, this implies the cycle class map from $A^p(X)$ to $H^{2p}_{\mathrm{Hdg}}(X)$ is surjective.

Idea of proof ...

- and

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^p(X) & \longrightarrow & A^p(\widetilde{Y}) & \longrightarrow & A^p(E \cap \widetilde{Y}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(X) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(\widetilde{Y}) & \longrightarrow & H^{2p}_{\mathrm{Hdg}}(E \cap \widetilde{Y}) \end{array}$$

- Suffices to show: the cycle class map from $A^p(\mathcal{Y}_0)$ to $H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0)$ is surjective i.e., $H^{2p}_A(\mathcal{Y}_0) = H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0)$.
- Since

$$0 \to H^{2p}_{\mathrm{Hdg}}(X) \to H^{2p}_{\mathrm{Hdg}}(\mathcal{Y}_0) \to H^{2p}_{\mathrm{Hdg}}(E). \tag{5}$$

is an exact sequence of $\mathbb{Q}$-vector spaces, this implies the cycle class map from $A^p(X)$ to $H^{2p}_{\mathrm{Hdg}}(X)$ is surjective.

Thank you for your attention !