Seshadri constants for parabolic bundles

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This talk is based on a joint work with Indranil Biswas, Krishna Hanumanthu and Nabanita Ray.

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- A line bundle L is called **ample** if some positive integral multiple $L^{\otimes m}$ of it is very ample.

Definition (Nef line bundle)

A line bundle L on a projective variety X is called nef if $L \cdot C = \deg(L|_C) \geq 0$ for every irreducible curve $C \subset X$.

Nef and Ample vector bundles

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For a vector bundle E on a projective variety X, we consider Grothendieck's projectivization $\mathbb{P}(E) := \mathrm{Proj}\big(\oplus_{i \geq 0} \mathrm{Sym}^i(E)\big)$.

Definition (Hartshorne)

A vector bundle E on a projective variety X is called ample (nef) if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (nef) on $\mathbb{P}(E)$.

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- An $\mathbb{R}$-divisor $D = \sum_i a_i D_i \in \mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes \mathbb{R}$ is called numerically trivial, denoted by $D \equiv 0$ if

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- An $\mathbb{R}$-divisor $D = \sum_i a_i D_i \in \mathsf{Div}(X)_{\mathbb{R}} := \mathsf{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ on X is called **nef** if

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for all irreducible curve $C \subseteq X$.

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for all irreducible curve $C \subseteq X$.

- The intersection product being independent of numerical equivalence class, one can talk about nef classes in $N^1(X)_{\mathbb{R}}$.

Nakai-Moishezon-Kleiman criterion

A line bundle L on a projective variety X is ample if and only if $L^{\dim V} \cdot V > 0$ for every positive-dimensional irreducible closed subvariety $V \subseteq X$ (including the irreducible components of X).

Numerical criterion for ampleness

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Seshadri's criterion

A line bundle L on a projective variety X is ample if and only if there exists a positive number $\epsilon > 0$ such that

$$\frac{(L \cdot C)}{\operatorname{mult}_x C} \geq \epsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x. Here $L \cdot C$ is the intersection product of the line bundle L with the curve C.

Seshadri constants for line bundles

Let X be a complex projective variety, and let L be a nef line bundle on X.

Definition (Demailly)

For a point $x \in X$, the **Seshadri constant** of L at x, denoted by $\varepsilon(X, L, x)$, is defined to be

$$\varepsilon(X, L, x) := \inf_{x \in C} \left\{ \frac{L \cdot C}{\mathrm{mult}_x C} \right\},$$

where the infimum is taken over all irreducible and reduced curves $C \subset X$ passing through x with multiplicity $\mathrm{mult}_x C$.

Alternatively, we can define Seshadri constants as follows. Let

$$\pi : \mathrm{Bl}_x X = \widetilde{X} \longrightarrow X$$

be the blow up of X at x, and let E denote the exceptional divisor.

$$\varepsilon(X, L, x) = \sup \{ \lambda \geq 0 \mid \pi^*(L) - \lambda E \text{ is nef} \}.$$

We consider a nef vector bundle E on a projective variety X and $x \in X$. Let us consider the following pullback diagram under the blow up map $\rho_x : \mathrm{Bl}_x(X) \longrightarrow X$

$$\begin{array}{ccc} \mathbb{P}(E) \times_X \widetilde{X}_x = \mathbb{P}\big(\rho_x^*(E)\big) & \xrightarrow{\widetilde{\rho_x}} & \mathbb{P}(E) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Bl}_x X & \xrightarrow{\rho_x} & X \end{array}$$

Let $\widetilde{\xi}_x$ be the numerical equivalence class of the tautological bundle $\mathcal{O}_{\mathbb{P}(\rho_x^* E)}(1)$, and $\widetilde{E}_x := \widetilde{\rho_x}^{-1}(F_x)$, where F_x is the class of the fibre of the map π over the point x.

DEFINITION (HACON)

The **Seshadri constant** of E at $x \in X$ is defined as

$$\varepsilon(X, E, x) := \sup\Big\{\lambda \in \mathbb{R}_{>0} \mid \widetilde{\xi}_x - \lambda \widetilde{E}_x \text{ is nef}\Big\}.$$

- Let X be a connected smooth complex projective variety of dimension d, and let $D \subset X$ be an effective divisor on X.

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- Let E be a torsion-free coherent $\mathcal{O}_X$-module. A **quasi-parabolic structure** on E with respect to D is a filtration by $\mathcal{O}_X$-coherent subsheaves

$$E = \mathcal{F}_1(E) \supset \mathcal{F}_2(E) \supset \cdots \supset \mathcal{F}_l(E) \supset \mathcal{F}_{l+1}(E) = E(-D),$$

where $E(-D) := E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$. The above integer l is called the **length** of the filtration.

Parabolic Vector bundles

- Let X be a connected smooth complex projective variety of dimension d, and let $D \subset X$ be an effective divisor on X.
- Let E be a torsion-free coherent $\mathcal{O}_X$-module. A **quasi-parabolic structure** on E with respect to D is a filtration by $\mathcal{O}_X$-coherent subsheaves

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  where $E(-D) := E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$. The above integer l is called the **length** of the filtration.
- A **parabolic structure** on E with respect to D is a quasi-parabolic structure as above together with a system of **weights** $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$, where each α_i is a real number such that $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1$. The numbers $\{\alpha_i\}_{i=1}^{l}$ are called **parabolic weights** and we say that α_i is attached to $\mathcal{F}_i(E)$.

Parabolic Vector Bundles

- A **parabolic vector bundle** with parabolic divisor D is a vector bundle E with a parabolic structure with respect to D.
- For any parabolic vector bundle E_* defined as above, and for any $t \in \mathbb{R}$, define the following filtration $\{E_t\}_{t\in\mathbb{R}}$ of coherent sheaves parametrized by $\mathbb{R}$:

$$E_t = \mathcal{F}_i(E)(-[t]D),$$

where $[t]$ is the integral part of t and $\alpha_{i-1} < t - [t] \le \alpha_i$ with $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. The filtration $\{E_t\}_{t\in\mathbb{R}}$ evidently determines the parabolic structure $\big(E, \mathcal{F}_*, \alpha_*\big)$ uniquely.

Consider the decomposition

$$D = \sum_{j=1}^{n} n_j D_j, \tag{1}$$

where every D_i is a reduced irreducible divisor, and $n_j \geq 1$. Let

$$f_j : n_j D_j \longrightarrow X$$

denote the inclusion map of the subscheme $n_j D_j$. For each $1 \leq j \leq n$, choose a filtration

$$0 = F^j_{l_j+1} \subset F^j_{l_j} \subset F^j_{l_j-1} \subset \cdots \subset F^j_1 = f_j^* E\,. \tag{2}$$

Fix real numbers α^j_k, $1 \leq k \leq l_j + 1$, such that

$$1 = \alpha^j_{l_j+1} > \alpha^j_{l_j} > \alpha^j_{l_j-1} > \cdots > \alpha^j_2 > \alpha^j_1 \geq 0.$$

For every $1 \leq j \leq n$ and $1 \leq k \leq l_j + 1$, define the coherent subsheaf $\overline{F}^i_j \subset E$ using the following short exact sequences:

$$0 \longrightarrow \overline{F}^j_k \longrightarrow E \longrightarrow (f_j^* E)/F^j_k \longrightarrow 0. \tag{3}$$

For $1 \leq j \leq n$ and $0 \leq t \leq 1$, let l^j_t be the smallest number in the set of integers

$$\{k \in \{1, 2, \cdots, l_j + 1\} \mid \alpha^j_k \geq t\}.$$

Finally, set

$$E_t := \bigcap_{j=1}^{n} \overline{F}^j_{l_{t-[t]}}((t - [t])D) \subseteq E((t - [t])D). \tag{4}$$

The filtration $\{E_t\}_{t \in \mathbb{R}}$ in (4) defines a parabolic structure on E. It is straightforward to check that all parabolic structures on E, with D as the parabolic divisor, arise this way.

Parabolic Vector bundles with rational coefficients

Henceforth we will always impose the following four conditions on the parabolic bundles E_*, with parabolic divisor D, that we will consider:

- **A** The parabolic divisor $D = \sum_{i=1}^{n} n_i D_i$ is a normal crossing divisor, i.e., all $n_i = 1$ and D_i are smooth divisors and they intersect transversally.
- **B** For each $1 \leq j \leq n$, choose a filtration

$$0 = F^j_{l_j+1} \subset F^j_{l_j} \subset F^j_{l_j-1} \subset \cdots \subset F^j_1 = f_j^* E\,. \tag{5}$$

  All F^i_j on D_i in are subbundles of $f_i^* E$ for every i.
- **C** All the weights α^i_j are rational numbers; so $\alpha^i_j = m^i_j / N$, where N is some fixed integer and $m^i_j \in \{0, 1, \cdots, N-1\}$.
- **D** Every point x of D has a neighborhood $U_x \subset X$, and a decomposition of $E\big|_{U_x}$ into a direct sum of line bundles, such that the filtration of all $E\big|_{U_x \cap D_i}$, $1 \leq i \leq n$, are constructed using the decomposition.

Associated Orbifold bundles

Let $\psi : \Gamma \longrightarrow \mathrm{Aut}(Y)$ be a finite group Γ acting on Y. An **orbifold bundle** on Y, with Γ as the **orbifold group** , is a vector bundle V on Y together with a lift of the action of Γ on Y to V, i.e. Γ acts on the total space of V such that the action of any $g \in \Gamma$ gives a vector bundle isomorphism between V and $\psi(g^{-1})^* V$.

Theorem (Biswas)

Let E_* be a parabolic bundle on X. Then using the Galois cover

$$\gamma : Y \longrightarrow X$$

with Galois group Γ, we can construct an orbifold bundle E' on Y such that the parabolic bundle E_* is recovered from it by taking Γ-invariants of the direct image of the twists of E' using the irreducible components of D.

Parabolic Ampleness

Definition (Biswas)

A parabolic bundle E_* is parabolic ample (respectively, nef) if and only if the corresponding orbifold bundle E' is ample (respectively, nef) as a vector bundle (in the sense of Hartshorne).

Parabolic Bundles as Ramified Bundles

Let X be a smooth complex projective variety, and let D be a normal crossing divisor on X. A **ramified principal** $\mathrm{GL}(n,\mathbb{C})$**-bundle over** X **with ramification over** D

$$\phi : E_{\mathrm{GL}(n,\mathbb{C})} \longrightarrow X$$

is a smooth complex quasiprojective variety equipped with an algebraic right action of $\mathrm{GL}(n,\mathbb{C})$

$$f : E_{\mathrm{GL}(n,\mathbb{C})} \times \mathrm{GL}(n,\mathbb{C}) \longrightarrow E_{\mathrm{GL}(n,\mathbb{C})}$$

satisfying the following five conditions:

- $\phi \circ f = \phi \circ p_1$, where p_1 is the natural projection of $E_{\mathrm{GL}(n,\mathbb{C})} \times \mathrm{GL}(n,\mathbb{C})$ to $E_{\mathrm{GL}(n,\mathbb{C})}$,
- for each point $x \in X$, the action of $\mathrm{GL}(n,\mathbb{C})$ on the reduced fiber $\phi^{-1}(x)_{red}$ is transitive,
- the restriction of ϕ to $\phi^{-1}(X - D)$ a principal $\mathrm{GL}(n,\mathbb{C})$-bundle over $X - D$,

- for each irreducible component $D_i \subset D$, the reduced inverse image $\phi^{-1}(D_i)_{red}$ is a smooth divisor and

$$\widehat{D} := \sum_{i=1}^{l} \phi^{-1}(D_i)_{red}$$

  is a normal crossing divisor on $E_{\mathrm{GL}(n,\mathbb{C})}$, and
- for any point $x \in D$, and any point $z \in \phi^{-1}(x)$, the isotropy subgroup $G_z \subset \mathrm{GL}(n,\mathbb{C})$, for the action of $\mathrm{GL}(n,\mathbb{C})$ on $E_{\mathrm{GL}(n,\mathbb{C})}$, is a finite group, and if x is a smooth point of D, then the natural action of G_z on the quotient line $T_z E_{\mathrm{GL}(n,\mathbb{C})}/T_z\phi^{-1}(D)_{\mathrm{red}}$ is faithful.

THEOREM (BALAJI, BISWAS AND NAGARAJ)

There is a natural bijective correspondence between the complex vector bundles of rank n on X and the principal $\mathrm{GL}(n, \mathbb{C})$-bundles on X. This bijection sends a principal $\mathrm{GL}(n, \mathbb{C})$-bundle F to the vector bundle $F \times^{\mathrm{GL}(n,\mathbb{C})} \mathbb{C}^n$ associated to F for the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. This correspondence extends to a bijective correspondence between the ramified principal $\mathrm{GL}(n, \mathbb{C})$-bundles with ramification over D and parabolic vector bundles of rank n with D as the parabolic divisor

Result (Biswas, Laytimi)

Let E_* be a parabolic vector bundle over X of rank n. Let

$$\phi \ : \ E_{\mathrm{GL}(n,\mathbb{C})} \longrightarrow X$$

be the corresponding ramified principal $\mathrm{GL}(n,\mathbb{C})$-bundle with ramification divisor D. Consider the standard action of $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}^n$; it induces an action of $\mathrm{GL}(n,\mathbb{C})$ on the projective space $\mathbb{P}^{n-1}$. The **projectivization** of E_*, denoted by $\mathbb{P}(E_*)$, is defined to be the associated (ramified) fiber bundle

$$\mathbb{P}(E_*) := E_{\mathrm{GL}(n,\mathbb{C})}(\mathbb{P}^{n-1}) := E_{\mathrm{GL}(n,\mathbb{C})} \times^{\mathrm{GL}(n,\mathbb{C})} \mathbb{P}^{n-1} \longrightarrow X.$$

Definition

Take any point $x \in D$ and any $z \in \phi^{-1}(x)$. Let n_x be the order of the finite group G_z. The number of distinct integers n_x as x varies over D is finite. Let

$$N(E_*) = \text{l.c.m.}\{n_x \mid x \in D\} \tag{6}$$

Consider the action of $\mathrm{GL}(n, \mathbb{C})$ on the total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))$ constructed using the standard action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. Let

$$E_{\mathrm{GL}(n,\mathbb{C})}(\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))) := E_{\mathrm{GL}(n,\mathbb{C})} \times^{\mathrm{GL}(n,\mathbb{C})} \mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*)) \longrightarrow X$$

be the associated fiber bundle. As the natural projection $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*)) \longrightarrow \mathbb{P}^{n-1}$ intertwines the actions of $\mathrm{GL}(n, \mathbb{C})$ on $\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))$ and $\mathbb{P}^{n-1}$, it produces a projection

$$\mathcal{O}_{\mathbb{P}(E_*)}(1) := E_{\mathrm{GL}(n,\mathbb{C})}(\mathcal{O}_{\mathbb{P}^{n-1}}(N(E_*))) \longrightarrow E_{\mathrm{GL}(n,\mathbb{C})}(\mathbb{P}^{n-1}) = \mathbb{P}(E_*). \tag{7}$$

Let E_* be a parabolic nef vector bundle on a smooth projective variety X. We fix a point $x \in X$ and and let

$$\psi_x : \mathrm{Bl}_x(X) \longrightarrow X$$

be the blow up of X at x with exceptional divisor $E_x = \psi_x^{-1}(x)$. Consider the following fiber product diagram:

$$\begin{array}{ccc} \mathrm{Bl}_{\rho^{-1}(x)}\big(\mathbb{P}(E_*)\big) = \mathbb{P}(E_*) \times_X \mathrm{Bl}_x(X) & \xrightarrow{\widetilde{\psi_x}} & \mathbb{P}(E_*) \\ \Big\downarrow \widetilde{\rho} & & \Big\downarrow \rho \\ \mathrm{Bl}_x(X) & \xrightarrow{\psi_x} & X \end{array}$$

The **parabolic Seshadri constant** of E_* at a point $x \in X$, denoted by $\varepsilon_*(E_*, x)$, is defined to be

$$\varepsilon_*(E_*, x) := \sup\left\{\lambda \in \mathbb{R}_{>0} \;\middle|\; \widetilde{\psi_x}^*(\xi) - \lambda \widetilde{\rho}^* E_x \text{ is nef}\right\}$$

where $\xi \equiv \mathcal{O}_{\mathbb{P}(E_*)}(1)$.

Alternative Characterization

Let E_* be a parabolic nef vector bundle on a smooth complex projective variety X, and let $x \in X$ be a point of X.

$$\rho : \mathbb{P}(E_*) \longrightarrow X$$

be the projectiviazation map. Let $\mathcal{C}_{\rho,x}$ be the set of all integral curves $C \subset \mathbb{P}(E_*)$ that intersect the fiber $\rho^{-1}(x)$ while not being contained in $\rho^{-1}(x)$. Then

$$\varepsilon_*(E_*, x) = \inf_{C \in \mathcal{C}_{\rho,x}} \left\{ \frac{\xi \cdot C}{\operatorname{mult}_x \rho_* C} \right\}.$$

Another Alternative Characterization

Let $E' \longrightarrow Y$ be the corresponding orbifold bundle over Y of a parabolic nef vector bundle E_* on X, where $\gamma : Y \longrightarrow X$ is a covering with Galois group $\Gamma = \mathsf{Gal}(\gamma)$. Consider the following fiber product diagram:

$$\begin{array}{ccccc} \mathbb{P}(E') \times_Y \mathsf{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\widetilde{\phi_x}} & \mathbb{P}(E') & \xrightarrow{\gamma'} & \mathbb{P}(E_*) = \mathbb{P}(E')/\Gamma \\ \Big\downarrow \widetilde{\tau} & & \Big\downarrow \tau & & \Big\downarrow \rho \\ \mathsf{Bl}_{\gamma^{-1}(x)} Y & \xrightarrow{\phi_x} & Y & \xrightarrow{\gamma} & X = Y/\Gamma. \end{array}$$

Let $E_{\gamma^{-1}(x)}$ be the exceptional divisor of the map ϕ_x and $\mathcal{O}_{\mathbb{P}(E')}(1) \equiv \xi'$.

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \sup\left\{\lambda \in \mathbb{R}_{>0} \mid \widetilde{\phi_x}^*(\xi') - \lambda \widetilde{\tau}^*(E_{\gamma^{-1}(x)}) \text{ is nef}\right\}$$

$$= N(E_*) \cdot \inf_{C \in \mathcal{C}_{\tau, \gamma^{-1}(x)}} \left\{ \frac{\xi' \cdot C}{\sum_{y \in \gamma^{-1}(x)} \mathsf{mult}_y \, \tau_* C} \right\}.$$

- E_* be a parabolic nef vector bundle on a smooth irreducible projective variety X. Then E_* is parabolic ample if and only if

$$\inf_{x \in X} \varepsilon_*(E_*, x) > 0,$$

where the infimum is taken over all points of X.

- Let E_* be a parabolic nef vector bundle on X, and $x \in X$ a point. Then

$$\varepsilon_*(E_*, x) \le \left(\frac{N(E_*)^{\dim \tau(W)} \xi'^{\dim W} \cdot [W]}{\binom{\dim W}{\dim \tau(W)} \cdot |\Gamma| (\xi'^{\dim W_{\gamma^{-1}(x)}} [W_{\gamma^{-1}(x)}])} \right)^{\frac{1}{\dim \tau(W)}},$$

as W ranges through the subvarieties of $\mathbb{P}(E')$ that meet $\tau^{-1}(\gamma^{-1}(x))$ without being contained in $\tau^{-1}(\gamma^{-1}(x))$.
In the above inequality, $W_{\gamma^{-1}(x)} := \tau^{-1}(\gamma^{-1}(x)) \bigcap W$.

Paraboli Seshadri constants of Parabolic nef vector bundles on smooth curves

For a parabolic vector bundle E_*, we define $\mu^{par}_{\min}(E_*)$ to be the parabolic slope of the minimal parabolic semistable subquotient of E_*. Note that if E' is the orbifold bundle on Y corresponding to E_* for the Galois morphism $\gamma : Y \longrightarrow X$ then we have

$$\mu_{\min}(E') = |\mathrm{Gal}(\gamma)| \cdot \mu^{par}_{\min}(E_*).$$

Theorem

Let E_* be a parabolic ample vector bundle over a smooth irreducible projective curve C with parabolic divisor D. Then for any point $x \in C$, the parabolic Seshadri constant satisfies the following:

$$\varepsilon_*(E_*, x) = N(E_*) \cdot \mu^{par}_{\min}(E_*) \text{ when } x \notin D, \text{ and}$$

$$\varepsilon_*(E_*, x) \geq N(E_*) \cdot \mu^{par}_{\min}(E_*) \text{ when } x \in D.$$

In particular, $\varepsilon_*(E_*, x) \geq \frac{N(E_*)}{\mathrm{rank}(E)}$ for every point $x \in C$.

Computing Seshadri constants by restricting to curves

Threorem

Let E_* be a parabolic nef vector bundle on a smooth irreducible complex projective variety X, and let $E' \longrightarrow Y$ be the corresponding orbifold bundle over Y. Then

$$\varepsilon_*(E_*, x) \;=\; N(E_*) \cdot \inf_{C \subset Y} \left\{ \frac{\mu_{\min}(\nu^* E')}{\sum_{y \in \gamma^{-1}(x)} \operatorname{mult}_y C} \right\},$$

where the infimum is taken over all irreducible curves $C \subset Y$ such that $C \bigcap \gamma^{-1}(x) \neq \emptyset$, and $\nu : \overline{C} \longrightarrow C$ is the normalization map.

Parabolic Seshadri constants for tensor product and symmetric power

Let E_* and F_* be two parabolic nef vector bundles on a smooth irreducible complex projective variety X having a common parabolic divisor $D \subset X$.

Theorem:

- For any positive integer m and for every point $x \in X$, we have

$$\varepsilon_*(S^m(E_*), x) = m\varepsilon_*(E_*, x)$$

- For every point $x \in X$

$$\varepsilon_*(E_* \otimes F_*, x) = N(E_* \otimes F_*) \cdot \left\{ \frac{\varepsilon_*(E_*, x)}{N(E_*)} + \frac{\varepsilon_*(F_*, x)}{N(F_*)} \right\}.$$

Thank You!